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## Problem I :

$$f(x)=\frac{x^3+2x^2}{2(x-1)^2}$$

1. Break $f(x)$ into the form $ax+b+\frac{cx+d}{2(x-1)^2}$ $(x\neq 1)$ ............ 4 pts

*By Euclidian division or by identification we get* $f(x)=\frac{1}{2}x+2+\frac{7x-4}{2(x-1)^2}$ *for* $x\neq 1$

2. Find the limits of $f$ at the ends of each interval of its definition set ............ 4 pts

$\lim_{x\to+\infty} f(x)=\lim_{x\to+\infty}\frac{x^3}{2x^2}=\lim_{x\to+\infty}\frac{x}{2}=+\infty$ *and similarly* $\lim_{x\to-\infty} f(x)=\lim_{x\to-\infty}\frac{x}{2}=-\infty$

*and* $\lim_{x\to1^\pm} f(x)=\lim_{x\to1^\pm}\frac{3}{2(x-1)^2}=+\infty$ *hence x = 1 is a"vertical " asymptote.*

3. Show that $(C_f)$ has a "vertical" and an oblique asymptote (Δ), give their equations. ............ 4 pts

from (1) we get $\lim_{x\to+\infty}[f(x)-(\frac{1}{2}x+2)]=\lim_{x\to+\infty}\frac{7x-4}{2(x-1)^2}=\lim_{x\to+\infty}\frac{7x}{2x^2}=\lim_{x\to+\infty}\frac{7}{2x}=0^+$

and similarly $\lim_{x\to-\infty}[f(x)-(\frac{1}{2}x+2)]=\lim_{x\to-\infty}\frac{7}{2x}=0^-$

4. Justify the position of $(C_f)$ with respect to this asymptote. ............ 2 pts

*From the sign of the above limits we can say that* $(C_f)$ *is above (Δ) in +∞ and under (Δ) in -∞*

5. Find the derivative $f'(x)$ and factor it in binomials ............ 6 pts

$f'(x)=\frac{(3x^2+4x)2(x-1)^2-(x^3+2x^2)4(x-1)}{4(x-1)^4}=\frac{x^3-3x^2-4x}{2(x-1)^3}=\frac{x(x+1)(x-4)}{2(x-1)^3}$ *for* $x\neq 1$

6. Give the zeroes of $f'(x)$ and justify the signs of the derivative. ............ 6 pts

*$f'(x)=0 \Leftrightarrow x=0$ or $x=-1$ or $x=4$ and Sign[$f'(x)$] = Sign $x(x-1)(x+1)(x-4)$ for $x\neq 1$*

7. Find the intersections of $(C_f)$ with the axes of coordinates. ............ 4 pts

*$f(x)=0 \Leftrightarrow x=0$ or $x=-2$ and $f(0)=0$*

8. Find the intersection of $(C_f)$ with its asymptote (Δ). ............ 2 pts

$f(x)=\frac{1}{2}x+2 \Leftrightarrow \frac{7x-4}{2(x-1)^2}=0 \Leftrightarrow x=\frac{4}{7}$ and then $y=\frac{1}{2}\frac{4}{7}+2=\frac{16}{7}\simeq 2.3$ Point I(0.4 ; 2.3).

9. Find the equation of the tangent line in $A(-2;0)$ to $(C_f)$ ............ 2 pts

*$f'(-2)=2/9$ there for the equation of $(T_A)$ is : $y=2/9\,(x+2)=2/9\,x+4/9$*

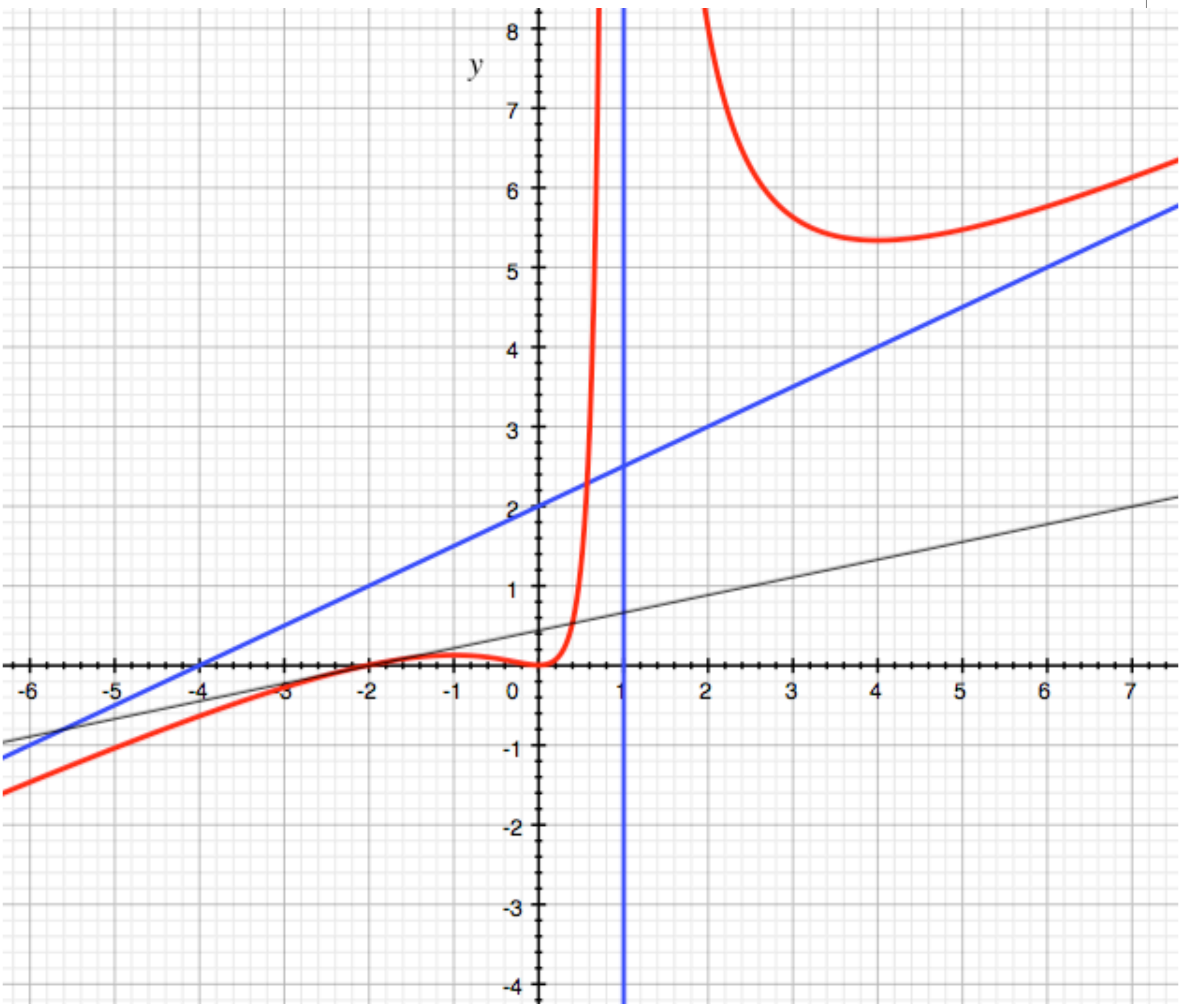
10. Show <u>all</u> the previous results in the chart below. ........................................................................ 6 pts

| $x$ | $-\infty$ | | $-2$ | | $-1$ | | $0$ | | $1$ | | $4$ | | $+$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| *Sign [f '(x)]* | | + | 0 | + | 0 | _ | 0 | + | \|\| | _ | 0 | + | |
| Variations *and **limits** of f* | $-\infty$ | ↗ | 0 | ↗ | M | ↘ | $m_1$ | ↗ | $+\infty$\|\|$+\infty$ | ↘ | $m_2$ | ↗ | $+\infty$ |

11. Give the approximate decimal value of the extremes here : ........................................................... 2 pts

$M = f(-1) = 1/8 = 0.125$ ; $m_1 = f(0) = 0$, *and* $m_2 = f(4) = 16/3 = 5.3$

12. Draw carefully $(C_f)$ its asymptotes, and the tangent line $(T_A)$. ........................................................ 8 pts



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Problem II : $$f(x)=\sqrt{\left|\frac{x^3+x^2-2}{x+2}\right|}$$

1. Break the function $f$ in two different functions $f_1$ and $f_2$ without absolute values, and specify their intervals of definition : ........ 4 pts

*$x^3+x^2-2=(x-1)(x^2+2x+2)$ therefore the sign of the fraction is that of $(x-1)(x+2)$ for $x\neq -2$*

$f_1(x)=\sqrt{\frac{x^3+x^2-2}{x+2}}$ $D_{f1}=]-\infty\,;-2[\cup[1\,;+\infty[$ ; $f_2(x)=\sqrt{\frac{-x^3-x^2+2}{x+2}}$ $D_{f2}=]-2\,;1]$

2. Find the limits of $f$ at the ends of each interval. ........ 4 pts

$$\lim_{x\to\pm\infty} f_1(x)=\lim_{x\to\pm\infty}\sqrt{\frac{x^3}{x}}=\lim_{x\to\pm\infty}\sqrt{x^2}=\lim_{x\to\pm\infty}|x|=+\infty$$

$\lim_{x\to-2^-} f_1(x)=+\infty$ and $\lim_{x\to-2^+} f_2(x)=+\infty$ because $f$ is positive and the limit of $x^3+x^2-2$ *is not 0 in -2*

$\lim_{x\to 1^\pm} f(x)=f(0)=0$ ( because f is *continuous* and defined in 0)

3. Show that $(C_f)$ has two oblique asymptotes $(\Delta_1)\ y=x-\frac{1}{2}$ & $(\Delta_2)\ y=-x+\frac{1}{2}$ ........ 4 pts

$$f_1(x)-(x-\frac{1}{2})=\sqrt{\frac{x^3+x^2-2}{x+2}}-(x-\frac{1}{2})=\frac{\frac{x^3+x^2-2}{x+2}-(x-\frac{1}{2})^2}{\sqrt{\frac{x^3+x^2-2}{x+2}}+(x-\frac{1}{2})}=\frac{7x-10}{4(x+2)\left[\sqrt{\frac{x^3+x^2-2}{x+2}}+(x-\frac{1}{2})\right]}$$

$$\lim_{x\to+\infty}\left[f_1(x)-(x-\frac{1}{2})\right]=\lim_{x\to+\infty}\frac{7}{4\left[\sqrt{\frac{x^3+x^2-2}{x+2}}+(x-\frac{1}{2})\right]}=\lim_{x\to+\infty}\frac{7}{8x}=0^+$$

$$f_1(x)-(-x+\frac{1}{2})=\sqrt{\frac{x^3+x^2-2}{x+2}}+x-\frac{1}{2}=\frac{\frac{x^3+x^2-2}{x+2}-(x-\frac{1}{2})^2}{\sqrt{\frac{x^3+x^2-2}{x+2}}-(x-\frac{1}{2})}=\frac{7x-10}{4(x+2)\left[\sqrt{\frac{x^3+x^2-2}{x+2}}-(x-\frac{1}{2})\right]}$$

$$\lim_{x\to-\infty}\left[f_1(x)-(-x+\frac{1}{2})\right]=\lim_{x\to+\infty}\frac{7}{4\left[\sqrt{\frac{x^3+x^2-2}{x+2}}-(x-\frac{1}{2})\right]}=\lim_{x\to-\infty}\frac{7}{(-8x)}=0^+$$

4. Justify the position of $(C_f)$ with respect to these two asymptotes ........ 2 pts

*from the signs of the above results, we can tell that $(C_f)$ is above its asymptote in $+\infty$ and in $-\infty$.*

5. Give the derivatives of the functions $f_1$ and $f_2$ on each interval where they are defined ........ 8 pts

*for $x\in\,]-\infty\,;-2[\,\cup\,]\,1\,;+\infty[$ we have* $f_1'(x)=\frac{2x^3+7x^2+4x+2}{2(x+2)^2\sqrt{\frac{x^3+x^2-2}{x+2}}}$ ($x\neq 1$ and $x\neq -2$)

*for $x\in\,]-\infty\,;]-2\,;1\,[$, we have* $f_2'(x)=\frac{-(2x^3+7x^2+4x+2)}{2(x+2)^2\sqrt{\frac{-x^3-x^2+2}{x+2}}}$ ($x\neq 1$ and $x\neq -2$)

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6. Show that the sign of the derivatives depends on $P(x) = 2x^3 + 7x^2 + 4x + 2$.............................. 6 pts
and show that there is only one zero α for P(x), with -3.0 < α < -2.9 *[Use back of page]*

*$P'(x) = 6x^2 + 14x + 4$ has two zeroes $x' = -2$ and $x'' = -1/3$. ∴ $P'(x) > 0$ for $x \le -2$ or $x \ge -1/3$ and $P'(x) \le 0$ for $-2 \le x \le -1/3$. ∴ $P(-2)$ is a Maximum and $P(-1/3)$ is a Minimum.*
*Finally, since $P(-1/3) = 1.4 > 0$ it means that $P(x)$ has only one zero α between -3 and -2.9.*
*P is strictly increasing on the interval [-3.0 ; -2.9] and changes sign : $P(-3) = -1 < 0$, $P(-2.9) \approx +0,5 > 0$. The variations of P show that $x \ge -2.9 \Rightarrow P(x) > 0$ . See Graph of P below.*
*Then we have proved that $f'(x)$ has one and only one zero $\alpha \approx -3$. and $f(\alpha) \approx 4.4$ and $f'(x)$ changes sign only once on the interval ] -∞ ; -2 [. This allows us to fill the chart of sign of f'x) below.*

7. Study the singularity of $(C_f)$ in $A(1;0)$ by studying the limits of the rate of growth of the function $f$ on both sides of A........................................................................................................ 6 pts

$$\lim_{x\to1^+}\frac{f_1(x)-f_1(1)}{x-1}=\lim_{x\to1^+}\frac{1}{x-1}\sqrt{\frac{x^3+x^2-2}{x+2}}=\lim_{x\to1^+}\sqrt{\frac{(x-1)(x^2+2x+2)}{(x-1)^2(x+2)}}=\lim_{x\to1^+}\sqrt{\frac{x^2+2x+2}{(x-1)(x+2)}}=\lim_{x\to1^+}\sqrt{\frac{5}{3(x-1)}}=+\infty$$

$$\lim_{x\to1^-}\frac{f_2(x)}{x-1}=\lim_{x\to1^-}\frac{1}{x-1}\sqrt{\frac{-x^3-x^2+2}{x+2}}=\lim_{x\to1^-}(-1)\sqrt{\frac{(1-x)(x^2+2x+2)}{(1-x)^2(x+2)}}=\lim_{x\to1^-}(-1)\sqrt{\frac{x^2+2x+2}{(1-x)(x+2)}}=\lim_{x\to1^-}(-1)\sqrt{\frac{5}{3(1-x)}}=-\infty$$

*This means that in A we have a vertical tangent of the 鸟 type.*

8. Show the previous results in the chart below, with the intersections of $(C_f)$ with the axes.......... 6 pts

| $x$ | - ∞ | | *α ≈ -3* | | -2 | | 0 | | 1 | | + ∞ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| *Sign [$f_1$'(x)]* | | - | 0 | + | \|\| | ////// | ////// | ////// | \|\| +∞ | | + |
| *Sign [$f_2$'(x)]* | ////// | ////// | ////// | ////// | \|\| | - | - | - | -∞ \|\| | ////// | ////// |
| *Sign [f'(x)]* | | - | 0 | + | \|\| | - | - | - | -∞ \|\| +∞ | + | |
| Variations *and* ***limits*** of $f$ | -∞ | ↘ | $M_1$ | ↗ | +∞ \|\| +∞ | ↘ | 1 | ↘ | 0 | ↗ | +∞ |

9. Give the approximate decimal value of the extremes here.: *$M_1 = f(\alpha) \approx f(-3) \approx 4.4$* ............................ 2 pts

10. Draw carefully $(C_f)$ its asymptotes, and the tangent line $(T_A)$. ........................................................ 8 pts