Model study of a function with absolute value and radicals with singular points

Reminders :

1°) Neither the Square Root function or the Absolute value function has a derivative in 0. Hence a function including an absolute value and / or a Square Root cannot have a "regular" derivative where the expression under them has a zero.

2°) If a function includes an expression with an Absolute Value, it must be separated in two different functions before applying the derivatives formulas.

3°) On a point where the derivative formulas do not apply, we must study directly the limits of the rate of growth at that point, separately on the right and on the left hand side.

4°) If the right and left limits are different

$$\lim_{x\to a^+}\frac{f(x)-f(a)}{x-a}=\alpha \quad \& \quad \lim_{x\to a^-}\frac{f(x)-f(a)}{x-a}=\beta \quad \& \quad \alpha\neq\beta$$

then the point *(a;f(a))* is an angular point : there are two half tangents. f '(a) does not exist.

5°) If the above limits are +∞ or -∞, then there is no derivative but there is a vertical tangent line on that point. The position of the curve on each side of the vertical tangent line depends on the sign of the limit.

$$f(x)=x\sqrt{\left|\frac{1-x}{1+x}\right|}$$

1. ***Set of definition*** *:* $D_f=\mathbb{R}\setminus\{-1\}=]-\infty\,;-1[\ \bigcup\ ]-1\,;+\infty[$

2. ***Limits*** *:* $\lim_{x\to\pm\infty}f(x)=\lim_{x\to\pm\infty}x=\pm\infty \quad \because \quad \lim_{x\to\pm\infty}\sqrt{\left|\frac{1-x}{1+x}\right|}=\sqrt{|-1|}=1$ *and* $\lim_{x\to-1}f(x)=-\infty \quad \because \quad \lim_{x\to-1}\sqrt{\left|\frac{1-x}{1+x}\right|}=+\infty$

3. ***Asymptotes :*** *we have 4 infinite branches, with one vertical asymptote x=-1. To find the oblique asymptotes we must find* $\lim_{x\to\infty}\frac{f(x)}{x}=a$*, then* $\lim_{x\to\infty}[f(x)-ax]=b$.

   In this case we find a = +1 (from the previous limits of the radical), and we have

   $$\lim_{x\to\infty}[f(x)-x]=\lim_{x\to\infty}x\left[\sqrt{\left|\frac{x-1}{1+x}\right|}-1\right]\to \underline{\text{undecided limit }\infty\times 0}$$

   To get rid of this undetermined form we must transform the expression by using it's conjugate and separating it in two expressions to eliminate the absolute value. Hence, for x >1 we get :

   $$\lim_{x\to+\infty}[f(x)-x]=\lim_{x\to+\infty}x\left[\sqrt{\frac{x-1}{1+x}}-1\right]=\lim_{x\to+\infty}x\frac{\frac{x-1}{1+x}-1}{\sqrt{\frac{x-1}{1+x}}+1}=\lim_{x\to+\infty}\frac{-2x}{(x+1)\left[\sqrt{\frac{x-1}{1+x}}+1\right]}=\lim_{x\to+\infty}\frac{-2x}{(x+1)[2]}=\lim_{x\to+\infty}\frac{-2x}{2x}=-1$$

   which proves that y = x -1 is the equation of the asymptote for x→ +∞.
   By a similar transformation we would get also $\lim_{x\to-\infty}[f(x)-x]=-1$. Therefore *y = x -1 is also the equation of the asymptote for x→-∞.*
   In this case the position of the curve with respect of this asymptote is the sign of the difference between f(x) and (x-1). This would drive us to longer calculations and we will assume that this difference is positive when x >1 and negative when x <-1.

4. ***Derivative :*** *according to the above reminders we must begin by separating the expression of f(x) in two different functions : let* $f_1(x)=x\sqrt{\frac{x-1}{1+x}}$ *for x > 1 or x < -1; and* $f_2(x)=x\sqrt{\frac{1-x}{1+x}}$ *for -1< x < 1.*
   Neither one of these two functions has a derivative at x = 1 (because √x is not derivable in 0).
   We find that for x > 1 or x < -1, $f_1'(x)=\frac{x^2+x-1}{\sqrt{(x-1)(x+1)^3}}$ *for -1< x <1* $f_2'(x)=\frac{-x^2-x+1}{\sqrt{(x-1)(x+1)^3}}$

In both cases the Sign of the derivative is the sign of its numerator ± $(x^2 + x - 1)$ of which the zeroes are $(-1+\sqrt{5})/2 \approx 0.6$ and $(-1-\sqrt{5})/2 \approx -1.6$. Hence we can chart the sign of the derivatives of f_1 and f_2 to get the variations of f accordingly :

5. ***Study of the Singular point (1; 0)*** : $\lim\limits_{x\to 1^+} \dfrac{f_1(x)-f_1(1)}{x-1} = \lim\limits_{x\to 1^+} \dfrac{x\sqrt{\dfrac{x-1}{1+x}}}{x-1} \to$ undecided form $\dfrac{0}{0}$.

To get rid of the undetermined form we must transform the expression by to eliminating the "trouble maker" quantity (x-1) … $\lim\limits_{x\to 1^+} \dfrac{f_1(x)-f_1(1)}{x-1} = \lim\limits_{x\to 1^+} \dfrac{x\sqrt{\dfrac{x-1}{1+x}}}{x-1} = \lim\limits_{x\to 1^+} \dfrac{x\dfrac{x-1}{1+x}}{(x-1)\sqrt{\dfrac{x-1}{1+x}}} = \lim\limits_{x\to 1^+} \dfrac{x}{\sqrt{x^2-1}} = +\infty$

In the same fashion we would find that $\lim\limits_{x\to 1^-} \dfrac{f_2(x)-f_2(1)}{x-1} = \lim\limits_{x\to 1^-} \dfrac{x\sqrt{\dfrac{1-x}{1+x}}}{x-1} = \lim\limits_{x\to 1^-} \dfrac{x\dfrac{1-x}{1+x}}{(x-1)\sqrt{\dfrac{1-x}{1+x}}} = \lim\limits_{x\to 1^-} \dfrac{-x}{\sqrt{x^2-1}} = -\infty$

That is to say that the curve is having a vertical tangent on (1;0) (马 type).

6. ***Chart of the variations of f :***

x	$-\infty$		-1.6		-1		0		0.6		1		$+\infty$
Sign [f_1'(x)]		+	0	-	‖						‖ $+\infty$	+	
Sign [f_2'(x)]					‖		+		0	-	$-\infty$ ‖		
Sign [f'(x)]		+	0	-	‖		+		0	-	$-\infty$ ‖ $+\infty$	+	
Variations *and* ***limits*** of *f*	$-\infty$	↗	M_1	↘	$-\infty$ ‖ $-\infty$			↗	M_2	↘	0	↗	$+\infty$

$M_1 = f(-1.6) \approx -3.3$; $M_2 = f(0.6) \approx 0.3$

7. ***Curve representing the graph of f*** : (the graph is the set of ordered pairs $(x ; y = f(x))$)