Model study of a function with absolute value and radicals with axial symmetry.

Reminders : 1°) A "vertical" line $x = a$ is an axis of symmetry for the graph of a function f if and only if $f(a + X) = f(a - X)$ for any X such that $a \pm X$ belongs to the set of definition of f.

2°) A point $I(a;b)$ is a center of symmetry for the graph of the function f if and only if : $(a + X) + f(a - X) = 2b$.

3°) Neither the Square Root function or the Absolute value function has a derivative in 0. Hence a function including an absolute value and / or a Square Root cannot have a "regular" derivative where the expression under them has a zero.

4°) If a function includes an expression with an Absolute Value, it must be separated in two different functions <u>before</u> applying the derivatives formulas.

$$f(x) = \frac{x^2 + 4x}{\sqrt{|x^2 + 4x + 1|}}$$

1. ***Set of definition*** *:* $D_f = \mathbb{R} \setminus \{x / x^2 + 4x + 1 = 0\} =]-\infty; -2-\sqrt{3}[\bigcup]-2-\sqrt{3}; -2+\sqrt{3}[\bigcup]-2+\sqrt{3}; +\infty[$

2. ***Limits :*** $\lim\limits_{x\to\pm\infty} f(x) = \lim\limits_{x\to\pm\infty} \frac{x^2\left(1+\frac{4}{x}\right)}{|x|\sqrt{\left|1+\frac{4}{x}+\frac{1}{x^2}\right|}} = \lim\limits_{x\to\pm\infty} \frac{x^2}{|x|} = \lim\limits_{x\to\pm\infty} |x| = +\infty$; $\lim\limits_{x\to-2\pm\sqrt{3}} f(x) = -\infty \because \lim\limits_{x\to-2\pm\sqrt{3}} (x^2+4x) = -1$ & $\lim\limits_{x\to-2\pm\sqrt{3}} \sqrt{|x^2+4x+1|} = 0^+$

3. ***Asymptotes :*** *we have 4 infinite branches, with two vertical asymptotes x =-2-√3 and x =-2+√3*

To find the oblique asymptotes we must find $\lim\limits_{x\to\infty} \frac{f(x)}{x} = a$, *then* $\lim\limits_{x\to\infty} [f(x) - ax] = b$.

In this case we have $\lim\limits_{x\to\pm\infty} \frac{f(x)}{x} = \lim\limits_{x\to\pm\infty} \frac{x}{|x|}$ *so that a = +1 when x→+∞ ; and a = -1 when x→-∞*

$$\lim_{x\to+\infty} [f(x) - x] = \lim_{x\to+\infty} \frac{(x^2+4x) - x\sqrt{x^2+4x+1}}{\sqrt{x^2+4x+1}} = \lim_{x\to+\infty} \frac{4x^3 + 15x^2}{\sqrt{x^2+4x+1}\left[(x^2+4x) + x\sqrt{x^2+4x+1}\right]}$$

$$= \lim_{x\to+\infty} \frac{4x^3\left(1+\frac{15}{4x}\right)}{x^3\sqrt{1+\frac{4}{x}+\frac{1}{x^2}}\left[(1+\frac{4}{x}) + \sqrt{1+\frac{4}{x}+\frac{1}{x^2}}\right]} = \lim_{x\to+\infty} \frac{4x^3}{2x^3} = 2$$

which proves that y = x +2 is the equation of the asymptote for x→ +∞.
In a similar way we could find that $\lim\limits_{x\to-\infty} [f(x) + x] = -2$ *and y =- x – 2 is the equation of the asymptote for x→ -∞. In this case the position of the curve with respect of these asymptotes is the sign of the difference between f(x) and | x + 2 |.This would drive us to longer calculations and we will assume that this difference is always negative, which proves that the curve will always be under the oblique asymptotes.*

4. ***Derivative :*** *according to the above reminders we must begin by separating the expression of f(x) in two different functions : let* $f_1(x) = \frac{x^2+4x}{\sqrt{x^2+4x+1}}$ *for x <-2-√3 or x >-2+√3.*

and $f_2(x) = \frac{x^2+4x}{\sqrt{-x^2-4x-1}}$ *for -2-√3 < x <-2+√3*

Neither one of these two functions has a derivative at x = -2±√3 (because √x is not derivable in 0).

We find that $f_1'(x) = \frac{(x+2)(x^2+4x+2)}{(x^2+4x+1)\sqrt{x^2+4x+1}}$ *for x <-2-√3 or x >-2+√3*

and $f_2'(x) = \frac{(x+2)(x^2+4x+2)}{(x^2+4x+1)\sqrt{-x^2-4x-1}}$ *for -2-√3 < x <-2+√3*

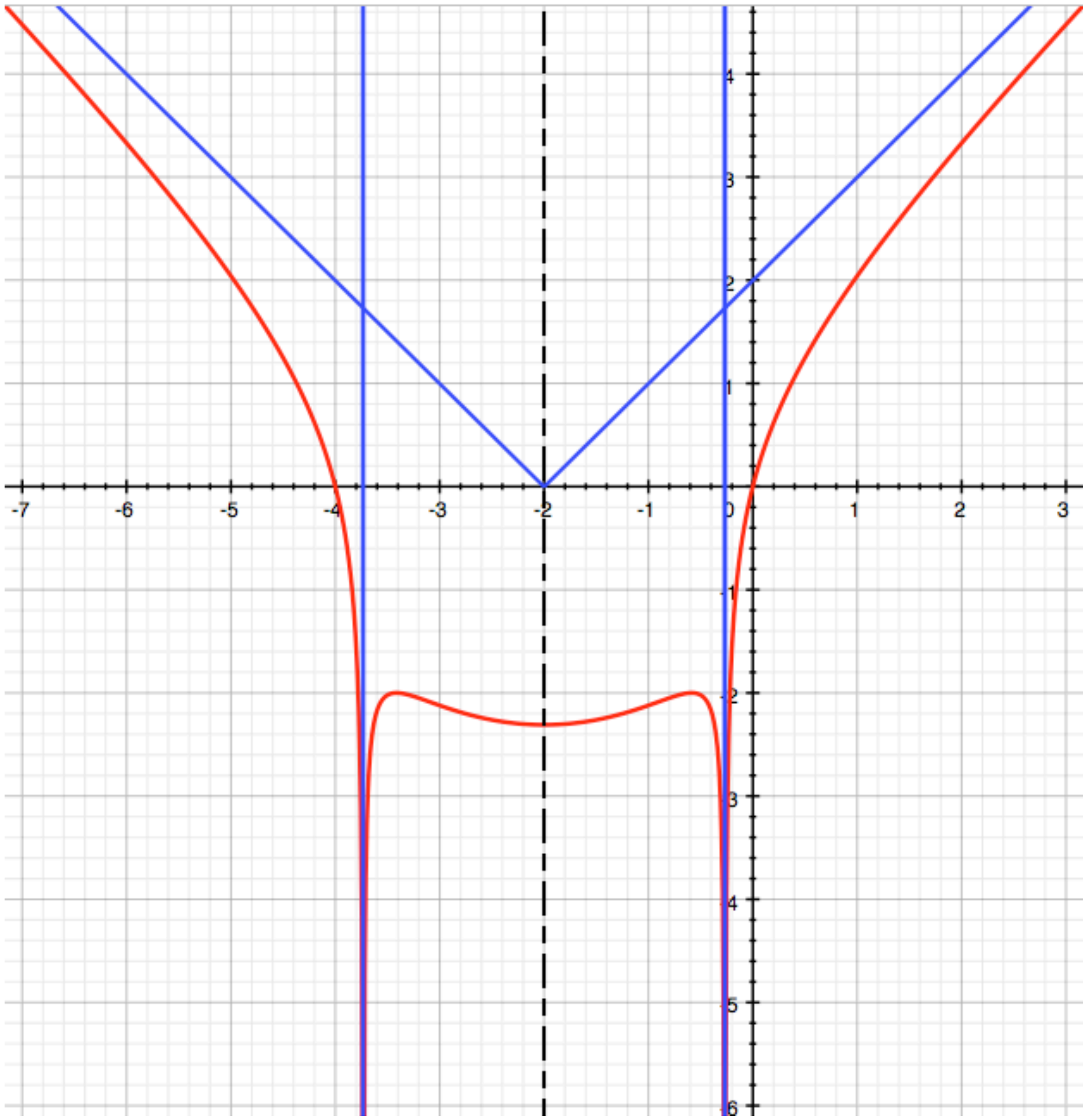
In both cases the Sign of the derivative is the sign of the product $(x+2)(x^2+4x+1)(x^2+4x+2)$ of which the zeroes are {-2 ; $(-2-\sqrt{3}) \approx -3.7$; $(-2+\sqrt{3}) \approx -0.3$; $(-2-\sqrt{2}) \approx -3.4$; $(-2+\sqrt{2}) \approx -0.6$ }. Hence we can chart the sign of the derivatives of f_1 and f_2 to get the variations of f accordingly :

5. ***Chart of the variations of f :***

x	$-\infty$		-3.7		-3.4		-2		-0.6		-0.3		$+\infty$
Sign $[f_1'(x)]$		-	||								||	+	
Sign $[f_2'(x)]$			||	+	0	-	0	+	0	-	||		
Sign $[f'(x)]$		-	||	+	0	-	0	+	0	-	||	+	
Variations and **limits** of f	$+\infty$	↘	$-\infty$||$-\infty$	↗	m_1	↘	m_2	↗	m_3	↘	$-\infty$||$-\infty$	↗	$+\infty$

$m_1 = f(-2-\sqrt{3}) = -2$; $m_2 = f(-2) \approx -2.3$; $m_3 = f(-2+\sqrt{3}) = -2$

7. ***Curve representing the graph of f*** : (the graph is the set of ordered pairs $(x ; y = f(x))$)



8. ***Symmetry :*** *the line x = -2 is an axis of symmetry because we have f(-2+X) = f(-2-X) for any X.*

$$f(-2+X) = \frac{X^2-4}{\sqrt{X^2-3}} = f(-2-X)$$