Model for a complete study of a rational function

Reminders : 1°) The **set of definition** of the function must be defined in terms of **Intervals**.

2°) The **limits** of a rational function when the variable goes to infinity is the same as the ratio of its higher degrees terms.

3°) The horizontal and vertical **asymptotes** are determined by the limits of *f* at on the boundaries of each interval of the set of definition.

4°) The **variations** of a function are determined by the **sign of it's derivative**. Therefore it's, most of the time, necessary to calculate the derivative, determine it's zeroes and study whether or not there is a change of sign. If there is no change of sign then it's not a **maximum** or **minimum**, it's an **inflexion** point.

5°) A line (D) of equation $y = ax + b$ is an **oblique asymptote** to the curve representing the graph of the function if and only if the **difference *[f(x) – (ax+b)]*** tends towards 0 when x goes to infinity. The **position of the curve** with respect of the line (D) must be determined by the **sign** of that difference.

6°) The **zeroes** of a function can be approximately determined by observing the **changes of sign** in a given interval : if *f* is a derivable (hence continuous) and monotonous function on [a ; b] and if $f(a).f(b) < 0$ then there is one and only one α in [a ; b] such that $f(\alpha) = 0$. Then if the interval [a ; b] is small, a and b can be considered as approximate values of α.

7°) Summarize all the previous studies in a **chart** showing the boundaries of the definition set, the zeroes of the function, the zeroes of its derivative, the values of maximum or minimum, and the limits.

8°) Eventually study the **symmetries** with respect to a given vertical axis or a given point.

9°) Find the equation of **tangent** line at the intersection of the curve of the function with the "vertical" axis. Study the case of singular points (vertical tangent, half tangent, inflexion point).

10°)Draw carefully the curve after having carefully placed the asymptotes, the zeroes, the maximum and minimum, and the intersections with the axis.

$$f(x)=\frac{2x^3-x^2+2}{(2x-3)^2}$$

1. *Set of definition :* $D_f = \mathbb{R}\setminus\{\frac{3}{2}\} =]-\infty;\frac{3}{2}[\bigcup]\frac{3}{2};+\infty[$

2. $\lim\limits_{x\to+\infty} f(x)=\lim\limits_{x\to+\infty}\left(\frac{2x^3}{4x^2}\right)=\lim\limits_{x\to+\infty}\frac{x}{2}=+\infty$; $\lim\limits_{x\to-\infty} f(x)=\lim\limits_{x\to-\infty}\left(\frac{2x^3}{4x^2}\right)=\lim\limits_{x\to-\infty}\frac{x}{2}=-\infty$; $\lim\limits_{x\to\frac{3}{2}} f(x)=+\infty$

Therefore there are 4 infinite branches, with one vertical asymptote x=3/2. To find the oblique asymptotes one can use two methods :

a) Break the given expression of f(x) into fractions such as $f(x)=ax+b+\dfrac{cx+d}{(2x-3)^2}$ *and find a,b,c,d, by identification or by Euclidian division of the polynomials.*

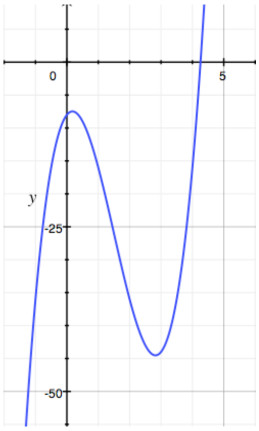
b) Calculate $\lim\limits_{x\to\infty}\dfrac{f(x)}{x}=a$*, then* $\lim\limits_{x\to\infty}[f(x)-ax]=b$*. But in both cases we need to know the sign of the difference [f(x)-(ax +b)] to be able to fix the position of the curve with respect to the asymptote (D) defined by y=ax+b. In this case we find* $f(x)=\frac{1}{2}x+\frac{5}{4}+\dfrac{\frac{21}{2}x-\frac{37}{4}}{(2x-3)^2}$ *=> asymptote equation* $y=\frac{1}{2}x+\frac{5}{4}$

because $\lim\limits_{x\to+\infty}\left[f(x)-\left(\frac{1}{2}x+\frac{5}{4}\right)\right]=\lim\limits_{x\to+\infty}\dfrac{\frac{21}{2}x-\frac{37}{4}}{(2x-3)^2}=\lim\limits_{x\to+\infty}\left[\frac{21x}{8x^2}\right]=\lim\limits_{x\to+\infty}\frac{21}{8x}=0^+(or\ 0^-\ for-\infty)$

Therefore (Cf) is above (D) for $x>\frac{37}{42}\simeq 0.8$ *and under (D) if* $x<\frac{37}{42}\simeq 0.8$ *(the change of sign corresponds to the point where the curve crosses it's asymptote).*

3. *Derivative* $f'(x) = \dfrac{4x^3 - 18x^2 + 6x - 8}{(2x-3)^3}$

4. *The Zeroes of the derivative are the Zeroes of* $P(x) = 4x^3 - 18x^2 + 6x - 8$. *Therefore we must study the variations of P to examine its zeroes and sign. To do so we must study the sign of* $P'(x) = 6(2x^2 - 6x + 1)$ *and chart the variations of P (The zeroes of P'(x) are* $\frac{3-\sqrt{7}}{2} \simeq 0.2$ *and* $\frac{3+\sqrt{7}}{2} \simeq 2.8$*)*



x	- ∞	0.2		2.8		α	+ ∞
Sign of [P '(x)]	+	0	-	**0**	+		
SIGN of P	-	-7.4	-	-44.5	-	**0**	+

From the graph we can see that α is approximately equal to 4.2, and we can check by calculations that P(4.2) <0 and P(4,3) >0 hence α ≈ 4.25

The sign of f'(x) is the same as the sign of P(x) for x > 1.5 and opposite for x < 1.5

x	- ∞ + ∞
Sign [f'(x)]	
Variations *and* ***limits*** of *f*