Euler 1707 - 1783

Euler's method of construction of the Exponential function

Objective : *build a function defined on R such that for any x Real,*

$$f'(x) = f(x) \text{ and } f(0) = 1$$

- Reminder : from the definition of the derivative of a function f in one point a :

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

$$\therefore \quad \frac{f(a+h) - f(a)}{h} = f'(a) + \varepsilon(h) \quad \text{with} \quad \lim_{h \to 0} \varepsilon(h) = 0$$

$$f(a+h) \approx f(a) + h.f'(a) + \quad h \cdot \varepsilon(h)$$

- *Then for any value of h, close to 0 (h <<1),* $h \cdot \varepsilon(h)$ *is negligeable*

$$f(a+h) \approx f(a) + h.f'(a)$$

In our case we have f'(a) = f(a) and for a =0, f(0) = 1

$$f(0+h) \approx f(0) + h.f'(0) = f(0) + h.f(0) = 1 + h$$

$$f(h) \approx 1 + h$$

$$\text{then } f(2h) = f(h+h) \approx f(h) + h.f'(h) = (1+h).f(h)$$

$$f(2h) \approx (1+h)^2$$

$$\text{then } f(3h) = f(2h+h) \approx f(2h) + h.f'(h) = (1+h)^2.f(h)$$

$$f(3h) \approx (1+h)^3$$

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$$f(nh) \approx (1+h)^n$$

$$f(nh) \approx (1+h)^n$$

for example let $h = 0.01$ *and* $n = 100$ *then*

$$f(1)=f(100 \times 0.01) \approx (1+0.01)^{100} \approx 2.7..$$

$$f(2)=f(200 \times 0.01) \approx (1+0.01)^{200} = [(1+0.01)^{100}]^2 \approx (2.7)^2 \approx 7.3..$$

$$f(3)=f(300 \times 0.01) \approx (1+0.01)^{300} = (2.7)^3 \approx 19.7..$$

then for any Integer k we would have :

$$f(k)=f(k \times 100 \times 0.01) \approx [(1+0.01)^{100}]^k \approx (2.7)^k$$

Let's call $e = \lim_{n \to \infty}\left(1+\frac{1}{n}\right)^n = 2.718...$

then for n = 100 we have $\left(1+\frac{1}{100}\right)^{100} \approx e \approx 2.7..$

therefore for any Integer k we have :

$$f(k)=f(k \times 100 \times 0.01) \approx [(1 + 0.01)^{100}]^k \approx e^k$$

If x is any Real number what can we write ?

Let's take n a number large enough so that h = x/n <0.01

$$f(x) = f(n \times \frac{x}{n}) \approx \left(1+\frac{x}{n}\right)^n = \left[\left(1+\frac{1}{\frac{n}{x}}\right)^{\frac{n}{x}}\right]^x \approx e^x$$

Problem : the formula :

$$\left(1+\frac{x}{n}\right)^n = \left[\left(1+\frac{1}{\frac{n}{x}}\right)^{\frac{n}{x}}\right]^x \approx e^x$$

is "*wrong*" …

because it was mathematically established that

$$\left(a^m\right)^n = a^{m\times n}$$

only for the Integers m and n

(not for Real numbers)

So what ? ! ? …

Euler *was not just able to calculate these values with a great acuracy, but he was mainly able to prove that the function defined by the original conditions* $f'(x) = f(x)$ *and* $f(0) = 1$ *is defined for any real and is continuous.*

(since it has a derivative).

He named it "Exponential" and he was also able to prove that such that new function Exp. had a the fundamental property of transforming sums into products for any real numbers u and v

$$Exp\,(u + v) = Exp(u) \cdot Exp(v)$$

Or

$$e^{u+v} = e^{u}.\ e^{v}$$

Then the previous formula makes sense.

More over Euler established a fundamental formula of development of functions in "power series" such that :

$$e^x = \frac{x^0}{0!} + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!} + \ldots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

It is this formula (using at least 6 or 7 terms) that is used in computers and pocket calculators to provide us with the "exact" values of Exp(x).

Then we can check that the values found by the differential relationship are correct approximate values.

h = 0,01

n	x	y = (1 + h)^n	z = Exp(x)
0	0	1	1
1	0,01	1,01	1,010050167
2	0,02	1,0201	1,02020134
3	0,03	1,030301	1,030454534
4	0,04	1,04060401	1,040810774
5	0,05	1,05101005	1,051271096
6	0,06	1,061520151	1,061836547
7	0,07	1,072135352	1,072508181
8	0,08	1,082856706	1,083287068
9	0,09	1,093685273	1,094174284
10	0,1	1,104622125	1,105170918
11	0,11	1,115668347	1,11627807
12	0,12	1,12682503	1,127496852
13	0,13	1,13809328	1,138828383
14	0,14	1,149474213	1,150273799
15	0,15	1,160968955	1,161834243
16	0,16	1,172578645	1,173510871
17	0,17	1,184304431	1,185304851
18	0,18	1,196147476	1,197217363
19	0,19	1,20810895	1,209249598
20	0,2	1,22019004	1,221402758
21	0,21	1,23239194	1,23367806
22	0,22	1,24471586	1,246076731
23	0,23	1,257163018	1,25860001
24	0,24	1,269734649	1,27124915
25	0,25	1,282431995	1,284025417
26	0,26	1,295256315	1,296930087
27	0,27	1,308208878	1,309964451
28	0,28	1,321290967	1,323129812
29	0,29	1,334503877	1,336427488
30	0,3	1,347848915	1,349858808
31	0,31	1,361327404	1,363425114
32	0,32	1,374940679	1,377127764
33	0,33	1,388690085	1,390968128
34	0,34	1,402576986	1,404947591
35	0,35	1,416602756	1,419067549
36	0,36	1,430768784	1,433329415
37	0,37	1,445076471	1,447734615
38	0,38	1,459527236	1,462284589
39	0,39	1,474122509	1,476980794
40	0,4	1,488863734	1,491824698
41	0,41	1,503752371	1,506817785
42	0,42	1,518789895	1,521961556
43	0,43	1,533977794	1,537257524
44	0,44	1,549317572	1,552707219
45	0,45	1,564810747	1,568312185
46	0,46	1,580458855	1,584073985
47	0,47	1,596263443	1,599994193
48	0,48	1,612226078	1,616074402
49	0,49	1,628348338	1,63231622
50	0,5	1,644631822	1,648721271
51	0,51	1,66107814	1,665291195
52	0,52	1,677688921	1,68202765
53	0,53	1,694465811	1,698932309
54	0,54	1,711410469	1,716006862
55	0,55	1,728524573	1,733253018
56	0,56	1,745809819	1,7506725
57	0,57	1,763267917	1,768267051
58	0,58	1,780900597	1,786038431
59	0,59	1,798709603	1,803988415
60	0,6	1,816696699	1,8221188
61	0,61	1,834863666	1,840431399
62	0,62	1,853212302	1,858928042
63	0,63	1,871744425	1,877610579
64	0,64	1,890461869	1,896480879
65	0,65	1,909366488	1,915540829
66	0,66	1,928460153	1,934792334
67	0,67	1,947744755	1,954237321
68	0,68	1,967222202	1,973877732
69	0,69	1,986894424	1,993715533
70	0,7	2,006763368	2,013752707
71	0,71	2,026831002	2,033991259
72	0,72	2,047099312	2,054433211
73	0,73	2,067570305	2,075080608
74	0,74	2,088246008	2,095935514
75	0,75	2,109128468	2,117000017
76	0,76	2,130219753	2,13827622
77	0,77	2,151521951	2,159766254
78	0,78	2,17303717	2,181472265
79	0,79	2,194767542	2,203396426
80	0,8	2,216715217	2,225540928
81	0,81	2,238882369	2,247907987
82	0,82	2,261271193	2,270499838
83	0,83	2,283883905	2,29331874
84	0,84	2,306722744	2,316366977
85	0,85	2,329789971	2,339646852
86	0,86	2,353087871	2,363160694
87	0,87	2,37661875	2,386910854
88	0,88	2,400384937	2,410899706
89	0,89	2,424388787	2,435129651
90	0,9	2,448632675	2,459603111
91	0,91	2,473119001	2,484322533
92	0,92	2,497850191	2,50929039
93	0,93	2,522828693	2,534509178
94	0,94	2,54805698	2,559981418
95	0,95	2,57353755	2,585709659
96	0,96	2,599272926	2,611696473
97	0,97	2,625265655	2,637944459
98	0,98	2,651518311	2,664456242
99	0,99	2,678033494	2,691234472
100	1	2,704813829	2,7182818