jiguanglaoshi@gmail.com - http://beijingshanmaths.org

∫ Integrals IBP and change of variables ∫

1. Show how to use the IBP formula to find the following integrals : *give u, u', v, v'…*

1. $F(x)=\int_1^x \ln t\,dt \quad (x>0)$ $\qquad u=\ln t \Rightarrow u'=\frac{1}{t}$; $v'=1 \Leftarrow v=t$ — 2 pts

$F(x)=\int_1^x \ln t.1.dt=\int_1^x u.v'.dt=[uv]_1^x-\int_1^x u'.v.dt=[t.\ln t]_1^x-\int_1^x 1.dt=[t\ln t-t]_1^x \Rightarrow \boxed{F(x)=x\ln x-x+1}$

2. $F(x)=\int_1^x (\ln t)^2\,dt \quad (x>0)$ $\qquad u=(\ln t)^2 \Rightarrow u'=\frac{2\ln t}{t}$; $v'=1 \Leftarrow v=t$ — 2 pts

$F(x)=\int_1^x (\ln t)^2.1.dt=\left[t.(\ln t)^2\right]_1^x-2\int_1^x \ln t.dt$

from the previous question we know that $\int_1^x \ln t.dt = x\ln x-x+1$

$F(x)=\left[t.(\ln t)^2\right]_1^x-2\int_1^x \ln t.dt=\left[t(\ln t)^2-2(t\ln t-t+1)\right]_1^x \Rightarrow \boxed{F(x)=x(\ln x)^2-2x\ln x+2x-2}$

3. $F(x)=\int_1^x t^2\ln t\,dt \quad (x>0)$ $\qquad u=\ln t \Rightarrow u'=\frac{1}{t}$; $v'=t^2 \Leftarrow v=\frac{t^3}{3}$ — 2 pts

$F(x)=\int_1^x t^2\ln t.\,dt=\left[\frac{t^3}{3}.\ln t\right]_1^x-\frac{1}{3}\int_1^x t^2.dt=\left[\frac{t^3}{3}.\ln t-\frac{t^3}{9}.\right]_1^x \Rightarrow \boxed{F(x)=\frac{x^3}{3}\ln x-\frac{x^3}{9}+\frac{1}{9}}$

4. $F(x)=\int_1^x \frac{\ln t}{t^2}dt \quad (x>0)$ $\qquad u=\ln t \Rightarrow u'=\frac{1}{t}$; $v'=\frac{1}{t^2} \Leftarrow v=-\frac{1}{t}$ — 2 pts

$F(x)=\int_1^x \ln t\frac{1}{t^2}dt=\left[-\frac{\ln t}{t}\right]_1^x+\int_1^x \frac{1}{t^2}dt=\left[-\frac{\ln t}{t}-\frac{1}{t}\right]_1^x \Rightarrow \boxed{F(x)=-\frac{\ln x}{x}-\frac{1}{x}+1}$

5. $F(x)=\int_1^x \frac{\ln t}{t^3}dt \quad (x>0)$ $\qquad u=\ln t \Rightarrow u'=\frac{1}{t}$; $v'=\frac{1}{t^3}=t^{-3} \Leftarrow v=\frac{t^{-2}}{-2}=-\frac{1}{2t^2}$ — 2 pts

$F(x)=\int_1^x \ln t\frac{1}{t^3}dt=\left[-\frac{\ln t}{2t^2}\right]_1^x+\frac{1}{2}\int_1^x \frac{1}{t^3}dt=\left[-\frac{\ln t}{2t^2}-\frac{1}{4t^2}\right]_1^x \Rightarrow \boxed{F(x)=-\frac{\ln x}{2x^2}-\frac{1}{4x^2}+\frac{1}{4}}$

2. Use twice the IBP formula to calculate :

1. $F(x)=\int_1^x \sin\ln t\,dt$ $\qquad u=\sin(\ln t) \Rightarrow u'=\frac{\cos(\ln t)}{t}$; $v'=1 \Leftarrow v=t$ — 5 pts

$F(x)=\int_1^x \sin\ln t.1.dt=\left[t\sin(\ln t)\right]_1^x-\int_1^x \cos(\ln t).1.dt$

$\Rightarrow F(x)=\int_1^x \sin\ln t.1.dt=\left[t\sin(\ln t)\right]_1^x-\left\{\left[t\cos(\ln t)\right]_1^x+\int_1^x \sin(\ln t)\,dt\right\}$

$\Rightarrow 2F(x)=\left[t\sin(\ln t)\right]_1^x-\left[t\cos(\ln t)\right]_1^x$

$\Rightarrow F(x)=\frac{1}{2}\left[t\sin(\ln t)-t\cos(\ln t)\right]_1^x \Rightarrow \boxed{F(x)=\frac{1}{2}\left[x\sin(\ln x)-x\cos(\ln x)+1\right]}$

2. $F(x)=\int_0^x t^2e^{-t}\,dt$ $\qquad u=t^2 \Rightarrow u'=2t$; $v'=e^{-t} \Leftarrow v=-e^{-t}$ — 5 pts

$F(x)=\left[-t^2e^{-t}\right]_1^x+2\int_0^x te^{-t}\,dt$ and $\int_0^x te^{-t}\,dt=\left[-te^{-t}\right]_0^x+\int_0^x e^{-t}\,dt=\left[-te^{-t}-e^{-t}\right]_0^x$

$F(x)=\left[-t^2e^{-t}\right]_0^x+2\left[-te^{-t}-e^{-t}\right]_0^x=\left[-t^2e^{t}-2te^{-t}-2e^{-t}\right]_0^x \Rightarrow \boxed{F(x)=-x^2e^{-x}-2xe^{-x}-2e^{-x}+2}$

3. Let $K_n = \int_{-1}^{1} (x^2-1)^n \, dx$.

i. $K_0 = \int_{-1}^{1} 1.dx = [x]_{-1}^{1} = 2$ $\quad K_1 = \int_{-1}^{1} (x^2-1).dx = \left[\frac{x^3}{3} - x\right]_{-1}^{1} = \left(\frac{1}{3}-1\right) - \left(\frac{-1}{3}-(-1)\right) = -\frac{4}{3}$ 2 pts

ii. Use IBP to prove that $K_n = -\frac{2n}{2n+1} K_{n-1}$ $\quad$ let $u = (x^2-1)^n \Rightarrow u' = 2nx(x^2-1)^{n-1}$ and $v' = 1 \Leftarrow v = x$ 3 pts

$$K_n = \int_{-1}^{1} (x^2-1)^n \, dx = \left[x(x^2-1)^n\right]_{-1}^{1} - 2n\int_{-1}^{1} x^2 (x^2-1)^{n-1} \, dx = 0 - 0 - 2n\int_{-1}^{1} (x^2-1+1)(x^2-1)^{n-1} \, dx$$

$$K_n = \int_{-1}^{1} (x^2-1)^n \, dx = -2n\int_{-1}^{1} (x^2-1)^n \, dx - 2n\int_{-1}^{1} (x^2-1)^{n-1} \, dx \Leftrightarrow (2n+1)K_n = -2nK_{n-1}$$

iii. Give a reduced formula for K_n in terms of n. 3 pts

$$K_n = \frac{(-2n)}{2n+1} K_{n-1} = \frac{(-2n)}{2n+1} \frac{(-2)(n-1)}{2(n-1)+1} K_{n-2} = \frac{(-2)n}{2n+1} \frac{(-2)(n-1)}{2(n-1)+1} \frac{(-2)(n-2)}{2(n-2)+1} K_{n-3}$$

$$\therefore K_n = \frac{(-1)^n 2n.2(n-1).2(n-2)..........4.2}{(2n+1).(2n-1).(2n-3).....3.1} I_0 = \frac{(-1)^n 2^{n+1}(n!)}{(2n+1).(2n-1).(2n-3).....3.1}$$

$$K_n = \frac{(-1)^n 2^{2n+1}[(n!)]^2}{(2n+1)!} \qquad K_n = \frac{(-1)^n 2^{2n+1}}{(2n+1)C_{2n}^n}$$

4. Use the simple change of variable $u = \cos x$ to calculate $I = \int_0^{\frac{\pi}{2}} (\cos x)^3 \sin x \, dx$ 3 pts

$$u = \cos x \Rightarrow du = -\sin x dx \Rightarrow \left\{ \begin{array}{l} x = \frac{\pi}{2} \Leftrightarrow u = 0 \\ x = 0 \Leftrightarrow u = 1 \end{array} \right\} \Rightarrow I = \int_1^0 -u^3 du = \int_0^1 u^3 du = \left[\frac{u^4}{4}\right]_0^1 \qquad \Leftrightarrow \quad I = \frac{1}{4}$$

5. Calculate $F(x) = \int_x^1 \ln\left(1+\frac{1}{t}\right) dt$ and $\lim_{x\to 0^+} F(x)$ 5 pts

Let $\left\{ \begin{array}{l} u = \ln\left(1+\frac{1}{t}\right) \Rightarrow u' = \frac{\frac{-1}{t^2}}{1+\frac{1}{t}} = \frac{-1}{t^2+t} \\ v' = 1 \quad \Leftarrow \quad v = t \end{array} \right\}$ then $F(x) = \left[t\ln\left(1+\frac{1}{t}\right)\right]_x^1 + \int_x^1 \frac{1}{1+t} dt$

$$F(x) = \int_x^1 \ln\left(1+\frac{1}{t}\right) dt = \left[t\ln\left(1+\frac{1}{t}\right) + \ln(1+t)\right]_x^1 \Rightarrow F(x) = 2\ln 2 - x\ln\left(1+\frac{1}{x}\right) - \ln(1+x)$$

$$0 < x \Rightarrow F(x) = 2\ln 2 - (x+1)\ln(x+1) + x\ln x \Rightarrow \lim_{x\to 0^+} F(x) = 2\ln 2 - 1\times\ln 1 + 0 \quad \Rightarrow \lim_{x\to 0^+} F(x) = 2\ln 2$$

6. $F(x) = \int_0^{\frac{1}{x}} \ln(1+t^2) \, dt$ and $\lim_{x\to+\infty} F(x)$

i. Give $\lim_{x\to+\infty} F(x)$ (justify your answer) : *the function defined by* $f(t)=\ln(1+t^2)$ is continuous on $\mathbb{R}$. 2 pts

therefore the function G defined by $G(x) = \int_0^X \ln(1+t^2) \, dt$ is continuous *[it's derivative is defined for any X, by G'(X) = f(X)]*, and the limit of *F(x)* is equal to the limit of G(X) :

$$x \to +\infty \Rightarrow X = \frac{1}{x} \to 0^+, \ \lim_{x\to+\infty} F(x) = \lim_{x\to+\infty} G\left(\frac{1}{x}\right) = \lim_{X\to 0} G(X) = G(0) = \int_0^0 \ln(1+t^2) \, dt = 0$$

ii. $F'(x) = (G[u(x)])' = G'[u(x)]\times u'(x) = f\left(\frac{1}{x}\right)\times\frac{-1}{x^2} = \ln\left(1+\frac{1}{x^2}\right)\left(\frac{-1}{x^2}\right)$ *3 pts*