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∫ Integrals IBP and change of variables ∫

1. Show how to use the IBP formula to find the following integrals : *give u, u', v, v'…*

1. $F(x)=\int_1^x \ln t\,dt \quad (x>0)$ $\quad u=\ln t \Rightarrow u'=\frac{1}{t} \quad ; \quad v'=1 \Leftarrow v=t$ 2 pts

$F(x)=\int_1^x \ln t.1.dt=\int_1^x u.v'.dt=[uv]_1^x-\int_1^x u'.v.dt=[t.\ln t]_1^x-\int_1^x 1.dt=[t\ln t-t]_1^x \Rightarrow \boxed{F(x)=x\ln x-x+1}$

2. $F(x)=\int_1^x (\ln t)^2\,dt \quad (x>0)$ $\quad u=(\ln t)^2 \Rightarrow u'=\frac{2\ln t}{t} \quad ; \quad v'=1 \Leftarrow v=t$ 2

$F(x)=\int_1^x (\ln t)^2.1.dt=\left[t.(\ln t)^2\right]_1^x-2\int_1^x \ln t.dt$

from the previous question we know that $\int_1^x \ln t.dt = x\ln x-x+1$

$F(x)=\left[t.(\ln t)^2\right]_1^x-2\int_1^x \ln t.dt=\left[t(\ln t)^2-2(t\ln t-t+1)\right]_1^x \Rightarrow \boxed{F(x)=x(\ln x)^2-2x\ln x+2x-2}$

3. $F(x)=\int_1^x t^2\ln t\,dt \quad (x>0)$ $\quad u=\ln t \Rightarrow u'=\frac{1}{t} \quad ; \quad v'=t^2 \Leftarrow v=\frac{t^3}{3}$ 2 pts

$F(x)=\int_1^x t^2\ln t.\,dt=\left[\frac{t^3}{3}.\ln t\right]_1^x-\frac{1}{3}\int_1^x t^2.dt=\left[\frac{t^3}{3}.\ln t-\frac{t^3}{9}.\right]_1^x \Rightarrow \boxed{F(x)=\frac{x^3}{3}\ln x-\frac{x^3}{9}+\frac{1}{9}}$

4. $F(x)=\int_1^x \frac{\ln t}{t^2}dt \quad (x>0)$ $\quad u=\ln t \Rightarrow u'=\frac{1}{t} \quad ; \quad v'=\frac{1}{t^2} \Leftarrow v=-\frac{1}{t}$ 2 pts

$F(x)=\int_1^x \ln t\frac{1}{t^2}dt=\left[-\frac{\ln t}{t}\right]_1^x+\int_1^x \frac{1}{t^2}dt=\left[-\frac{\ln t}{t}-\frac{1}{t}\right]_1^x \Rightarrow \boxed{F(x)=-\frac{\ln x}{x}-\frac{1}{x}+1}$

5. $F(x)=\int_1^x \frac{\ln t}{t^3}dt \quad (x>0)$ $\quad u=\ln t \Rightarrow u'=\frac{1}{t} \quad ; \quad v'=\frac{1}{t^3}=t^{-3} \Leftarrow v=\frac{t^{-2}}{-2}=-\frac{1}{2t^2}$ 2 pts

$F(x)=\int_1^x \ln t\frac{1}{t^3}dt=\left[-\frac{\ln t}{2t^2}\right]_1^x+\frac{1}{2}\int_1^x \frac{1}{t^3}dt=\left[-\frac{\ln t}{2t^2}-\frac{1}{4t^2}\right]_1^x \Rightarrow \boxed{F(x)=-\frac{\ln x}{2x^2}-\frac{1}{4x^2}+\frac{1}{4}}$

2. Use twice the IBP formula to calculate :

1. $F(x)=\int_1^x \cos\ln t\,dt$ $\quad u=\cos(\ln t) \Rightarrow u'=-\frac{\sin(\ln t)}{t} \quad ; \quad v'=1 \Leftarrow v=t$ 5 pts

$F(x)=\int_1^x \cos\ln t.1.\,dt=[t\cos(\ln t)]_1^x+\int_1^x \sin(\ln t).1.\,dt$

$\Rightarrow F(x)=\int_1^x \cos\ln t.1.\,dt=[t\cos(\ln t)]_1^x+[t\sin(\ln t)]_1^x-\int_1^x \cos(\ln t)\,dt$

$\Rightarrow 2\,F(x)=[t\cos(\ln t)]_1^x+[t\sin(\ln t)]_1^x$

$\Rightarrow F(x)=\frac{1}{2}[t\cos(\ln t)+t\sin(\ln t)]_1^x \Rightarrow \boxed{F(x)=\frac{1}{2}[x\cos(\ln x)+x\sin(\ln x)-1]}$

2. $F(x)=\int_0^x t^2e^t\,dt$ $\quad u=t^2 \Rightarrow u'=2t \quad ; \quad v'=e^t \Leftarrow v=e^t$ 5 pts

$F(x)=\left[t^2e^t\right]_1^x-2\int_0^x te^t\,dt$ and $\int_0^x te^t\,dt=\left[te^t\right]_0^x-\int_0^x e^t\,dt=\left[te^t-e^t\right]_0^x$

$F(x)=\left[t^2e^t\right]_0^x-2\left[te^t-e^t\right]_0^x=\left[t^2e^t-2te^t+2e^t\right]_0^x \Rightarrow \boxed{F(x)=x^2e^x-2xe^x+2e^x-2}$

3. Let $K_n = \int_{-1}^{1}(1-x^2)^n\,dx$.

i. $K_0 = \int_{-1}^{1} 1.dx = [x]_{-1}^{1} = 2$ $K_1 = \int_{-1}^{1}(1-x^2).dx = \left[x - \frac{x^3}{3}\right]_{-1}^{1} = \left(1-\frac{1}{3}\right) - \left(-1 - \frac{-1}{3}\right) = \frac{4}{3}$ 2 pts

ii. Use IBP to prove that $K_n = \frac{2n}{2n+1}K_{n-1}$ *let* $u = \left(1-x^2\right)^n \Rightarrow u' = n(-2x)\left(1-x^2\right)^{n-1}$ and $v' = 1 \Leftarrow v = x$ 3 pts

$$K_n = \int_{-1}^{1}(1-x^2)^n\,dx = \left[x(1-x^2)^n\right]_{-1}^{1} - 2n\int_{-1}^{1}(-x^2)(1-x^2)^{n-1}\,dx = 0 - 0 - 2n\int_{-1}^{1}(1-x^2-1)(1-x^2)^{n-1}\,dx$$

$$K_n = \int_{-1}^{1}(1-x^2)^n\,dx = -2n\int_{-1}^{1}(1-x^2)^n\,dx + 2n\int_{-1}^{1}(1-x^2)^{n-1}\,dx \Leftrightarrow \quad (2n+1)K_n = 2nK_{n-1}$$

iii. Give a reduced formula for K_n in terms of n. 3 pts

$$K_n = \frac{2n}{2n+1}K_{n-1} = \frac{2n}{2n+1}\frac{2(n-1)}{2(n-1)+1}K_{n-2} = \frac{2n}{2n+1}\frac{2(n-1)}{2(n-1)+1}\frac{2(n-2)}{2(n-2)+1}K_{n-3}$$

$$\therefore K_n = \frac{2n.2(n-1).2(n-2)..........4.2}{(2n+1).(2n-1).(2n-3).....3.1}K_0 = \frac{2^{n+1}(n!)}{(2n+1).(2n-1).(2n-3).....3.1}$$

$$K_n = \frac{2\left[2^n(n!)\right]^2}{(2n+1)!} \qquad K_n = \frac{2^{2n+1}}{(2n+1)C_{2n}^{n}}$$

4. Use the simple change of variable $u = \sin x$ to calculate $I = \int_0^{\frac{\pi}{2}}(\sin x)^3\cos x\,dx$ 3 pts

$$u = \sin x \Rightarrow du = \cos x\,dx \Rightarrow \left\{\begin{array}{l} x = \frac{\pi}{2} \Leftrightarrow u = 1 \\ x = 0 \Leftrightarrow u = 0 \end{array}\right\} \Rightarrow I = \int_0^1 u^3\,du = \left[\frac{u^4}{4}\right]_0^1 \quad \Leftrightarrow \quad I = \frac{1}{4}$$

5. Calculate $F(x) = \int_x^1 \ln\left(1+\frac{1}{t}\right)dt \quad and \quad \lim_{x\to 0^+} F(x)$ 5 pts

$$Let \left\{\begin{array}{l} u = \ln\left(1+\frac{1}{t}\right) \Rightarrow u' = \dfrac{\frac{-1}{t^2}}{1+\frac{1}{t}} = \dfrac{-1}{t^2+t} \\ v' = 1 \qquad \Leftarrow \qquad v = t \end{array}\right\} \text{ then } F(x) = \left[t\ln\left(1+\frac{1}{t}\right)\right]_x^1 + \int_x^1 \frac{1}{1+t}\,dt$$

$$F(x) = \int_x^1 \ln\left(1+\frac{1}{t}\right)dt = \left[t\ln\left(1+\frac{1}{t}\right) + \ln(1+t)\right]_x^1 \quad \Rightarrow \quad F(x) = 2\ln 2 - x\ln\left(1+\frac{1}{x}\right) - \ln(1+x)$$

$$0 < x \Rightarrow F(x) = 2\ln 2 - (x+1)\ln(x+1) + x\ln x \Rightarrow \lim_{x\to 0^+} F(x) = 2\ln 2 - 1\times\ln 1 + 0 \quad \Rightarrow \lim_{x\to 0^+} F(x) = 2\ln 2$$

6. $F(x) = \int_0^{\frac{1}{x}} \ln\left(1+t^2\right)dt \quad and \quad \lim_{x\to+\infty} F(x)$

i. Give $\lim_{x\to+\infty} F(x)$ (justify your answer) : *the function defined by f(t)=ln(1+t²)* is continuous on $\mathbb{R}$. 2 pts

therefore the function G defined by $G(x) = \int_0^X \ln\left(1+t^2\right)dt$ is continuous *[it's derivative is defined for any X, by G'(X) = f(X)]*, and the limit of *F(x)* is equal to the limit of G(X) :

$$x \to +\infty \Rightarrow X = \frac{1}{x} \to 0^+,\ \lim_{x\to+\infty} F(x) = \lim_{x\to+\infty} G\left(\frac{1}{x}\right) = \lim_{X\to 0} G(X) = G(0) = \int_0^0 \ln\left(1+t^2\right)dt = 0$$

ii. $F'(x) = \left(G[u(x)]\right)' = G'[u(x)]\times u'(x) = f\left(\frac{1}{x}\right)\times\frac{-1}{x^2} = \ln\left(1+\frac{1}{x^2}\right)\left(\frac{-1}{x^2}\right)$ *3 pts*