Study of the function f defined by f(0) =1 and for $x \neq 0$ by $f(x)=\left(1+\frac{4}{x^2}\right)^{\frac{x}{2}}$

I - 1. Complete the formula : $\lim_{x\to\pm\infty} x\ln\left(1+\frac{1}{x}\right)= 1$ and write the **complete proof** below 3pts

$$x.\ln\left(1+\frac{1}{x}\right)=\frac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{x}}\ ;\ X=\frac{1}{x}\to 0\ \therefore\ \lim_{X\to 0}\frac{\ln(1+X)}{X}=\lim_{X\to 0}\frac{\ln(1+X)-\ln(1)}{X}=\ln'(1)=\frac{1}{1}=1$$

2. Use the previous result to find $\lim_{x\to\pm\infty} x\ln\left(1+\frac{1}{x^2}\right)= 0^{\pm}$ with **complete proof** below 2pts

$$x.\ln\left(1+\frac{1}{x^2}\right)=\frac{1}{x}\left[x^2\ln\left(1+\frac{1}{x^2}\right)\right]\ ;\ \textit{and from } [1]\ \lim_{X\to+\infty}X\ln\left(1+\frac{1}{X}\right)=1\ \therefore\ \lim_{X\to\pm\infty}x.\ln\left(1+\frac{1}{x^2}\right)=0^{\pm}\times 1=0^{\pm}$$

3. Complete the formula : $\lim_{x\to 0^+} x\ln x = 0^-$ with **complete proof** below : 2pts

$$x.\ln(x)=-\frac{\ln\left(\frac{1}{x}\right)}{\frac{1}{x}}\ ;\ X=\frac{1}{x}\to+\infty\ ;\frac{\ln X}{X}\to 0^+\Rightarrow \lim_{x\to 0^+}x\ln x = 0^-$$

4. Use the previous result to find $\lim_{x\to 0} x\ln\left(1+\frac{1}{x^2}\right)= 0$ with **complete proof** below : 3pts

$$x.\ln\left(1+\frac{1}{x^2}\right)=x\ln(x^2+1)-2x\ln|x|\ \therefore\ \lim_{x\to 0}x.\ln\left(1+\frac{1}{x^2}\right)=0\times\ln(1)\pm 2\times 0=0$$

II - Let, for $x \neq 0$, $U(x)=x\ln\left(1+\frac{4}{x^2}\right)$.

Calculate the derivative (x≠0), $U'(x)=\ln\left(1+\frac{4}{x^2}\right)-\frac{8}{x^2+4}$ 2pts

Let $V(x) = U'(x)$, (x≠0) **calculate** $V'(x)=-\frac{8}{x(x^2+4)}+\frac{16x}{(x^2+4)^2}=8\frac{x^2-4}{x(x^2+4)^2}$: 2pts

Chart the **sign** of $V'(x)$ and the **variations** of V on R* with the **limits** and the **Minima** values. Explain why V(x) has two **zeroes** a and b and place them. 6pts

V is stictly monotonous and continuous on [-2; 0[and changes sign from m=V(-2)=ln2-1≈-.3 <0 to +∞, ∴ there is one and only one zero of V on [-2;0[, Same on]0;2].

x	- ∞		-2		a		0		b		2		+ ∞
Sign [V'(x)]		—	0		⊕		‖		—		0	⊕	
Variations and limits of V(x)	0	↘	- 0.3	↗	0	↗	‖	↘	0	↘	- 0.3	↗	0
Sign of U'(x)		—			0	⊕	‖	⊕	0		—		

III - Study the **variations of f** :

1. Show that she sign of *f'(x)* is the same as that of *U'(x)* *1 pt*

$$f(x)=\left(1+\frac{4}{x^2}\right)^{\frac{x}{2}}=\exp\left\{\ln\left[\left(1+\frac{4}{x^2}\right)^{\frac{x}{2}}\right]\right\}=\exp\frac{x}{2}\ln\left(1+\frac{4}{x^2}\right)=\exp\left(\frac{1}{2}U(x)\right)$$

$$\therefore f'(x)=\exp\left(\frac{1}{2}U(x)\right)\frac{1}{2}U'(x)=\frac{1}{2}\left(1+\frac{4}{x^2}\right)^{\frac{x}{2}}U'(x) \quad \therefore \quad Sgn[f'(x)]=Sgn[U'(x)]$$

2. Study the **limits of U(x) and f(x)** : *[indicate which formula is used]*

   a. Show the limit of *U(x)* and give the limit of *f(x)* at *x = 0* 2pts

$$\textit{from } [4]: \lim_{x\to 0} x\ln\left(1+\frac{1}{x^2}\right)= 0 \Rightarrow \lim_{x\to 0} 2\frac{x}{2}\ln\left(1+\frac{4}{x^2}\right)=2\times 0=0$$

$$\text{exp is continuous, then } \lim_{X\to 0}\exp(X)=\exp(0)=e^0=1 \therefore \lim_{x\to 0} f(x)=1$$

   b. Show the limits of *U(x)* and give the limit of *f(x)* in +∞ and -∞ 2pts

$$\textit{from } [2]: \lim_{x\to\pm\infty} x\ln\left(1+\frac{1}{x^2}\right)= 0 \Rightarrow \lim_{x\to\pm\infty} 2\frac{x}{2}\ln\left(1+\frac{4}{x^2}\right)=2\times 0=0$$

$$\text{exp is continuous, then } \lim_{X\to 0}\exp(X)=\exp(0)=e^0=1 \therefore \lim_{x\to 0} f(x)=1$$

3. Complete the **chart** of f with all the previous results : 2pts
   [one can admit that a© -1 ; b ≈ 1 ; f(-1) ≈ 0.4 ; f(1) ≈ 2.2]

x	- ∞		-2		-1		0		1		2		+ ∞
***Sign** [f'(x)]*			—		0	⊕	‖	⊕	0		—		
***Variations and limits** of f(x)*	1	↘	0.5	↗	0.4	↗	①	↗	2.2	↘	2	↘	1

4. **Graph** of the function f *(show the asymptotes if any)* 3pts