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Study of the function f defined by f(0) =1 and for $x \neq 0$ by $f(x)=\left(1+\dfrac{1}{x^2}\right)^x$

I - 1. Complete the formula : $\lim\limits_{x\to\pm\infty} x\ln\left(1+\dfrac{1}{x}\right) = 1$ and write the **complete proof** below — 3pts

$$x.\ln\left(1+\frac{1}{x}\right)=\frac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{x}}\ ;\ X=\frac{1}{x}\to 0\ \therefore\ \lim_{X\to 0}\frac{\ln(1+X)}{X}=\lim_{X\to 0}\frac{\ln(1+X)-\ln(1)}{X}=\ln'(1)=\frac{1}{1}=1$$

2. Use the previous result to find $\lim\limits_{x\to\pm\infty} x\ln\left(1+\dfrac{1}{x^2}\right) = 0^{\pm}$ with **complete proof** below — 2pts

$$x.\ln\left(1+\frac{1}{x^2}\right)=\frac{1}{x}\left[x^2\ln\left(1+\frac{1}{x^2}\right)\right]\ ;\ \textit{and from } [1]\ \lim_{X\to+\infty} X\ln\left(1+\frac{1}{X}\right)=1\ \therefore\ \lim_{X\to\pm\infty} x.\ln\left(1+\frac{1}{x^2}\right)=0^{\pm}\times 1=0^{\pm}$$

3. Complete the formula : $\lim\limits_{x\to 0^+} x\ln x = 0^-$ with **complete proof** below : — 2pts

$$x.\ln(x)=-\frac{\ln\left(\frac{1}{x}\right)}{\frac{1}{x}}\ ;\ X=\frac{1}{x}\to+\infty\ ;\ \frac{\ln X}{X}\to 0^+\Rightarrow \lim_{x\to 0^+} x\ln x = 0^-$$

4. Use the previous result to find $\lim\limits_{x\to 0} x\ln\left(1+\dfrac{1}{x^2}\right) = 0$ with **complete proof** below : — 3pts

$$x.\ln\left(1+\frac{1}{x^2}\right)=x\ln(x^2+1)-2x\ln|x|\ \therefore\ \lim_{x\to 0} x.\ln\left(1+\frac{1}{x^2}\right)=0\times\ln(1)\pm 2\times 0=0$$

II - Let, for $x \neq 0$, $U(x)=x\ln\left(1+\dfrac{1}{x^2}\right)$.

Calculate the derivative (x≠0), $U'(x)=\ln\left(1+\dfrac{1}{x^2}\right)-\dfrac{2}{x^2+1}$ — 2pts

Let $V(x) = U'(x)$, (x≠0) **calculate** $V'(x)=-\dfrac{2}{x(x^2+1)}+\dfrac{4x}{(x^2+1)^2}=2\dfrac{x^2-1}{x(x^2+1)^2}$: — 2pts

Chart the **sign** of $V'(x)$ and the **variations** of V on R* with the **limits** and the **Minima** values. Explain why V(x) has two **zeroes** a and b and place them. — 6pts

V is stictly monotonous and continuous on [-1; 0[and changes sign from m=V(-1)=ln2-1≈-.3 <0 to +∞, ∴ there is one and only one zero of V on [-1;0[, Same on]0;1].

x	$-\infty$		-1		a		0		b		1		$+\infty$
Sign [V'(x)]		—	0		⊕		||		—		0	⊕	
Variations and limits of V(x)	0	↘	- 0.3	↗	0	↗	||	↘	0	↘	- 0.3	↗	0
Sign of U'(x)			—		0	⊕	||	⊕	0		—		

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III - Study the **variations of f** :

1. Show that she sign of *f'(x)* is the same as that of *U'(x)* *1 pt*

$$f(x)=\left(1+\frac{1}{x^2}\right)^x=\exp\left\{\ln\left[\left(1+\frac{1}{x^2}\right)^x\right]\right\}=\exp x\ln\left(1+\frac{1}{x^2}\right)=\exp(U(x))$$

$$\therefore f'(x)=\exp(U(x))U'(x)=\left(1+\frac{1}{x^2}\right)^x U'(x) \quad \therefore \quad Sgn[f'(x)]=Sgn[U'(x)]$$

2. Study the **limits of U(x) and f(x)** : *[indicate which formula is used]*

a. Show the limit of *U(x)* and give the limit of *f(x)* at *x = 0* 2pts

$$\textit{from } [4]: \lim_{x\to 0} x\ln\left(1+\frac{1}{x^2}\right)=0$$

exp is continuous, then $\lim_{X\to 0}\exp(X)=\exp(0)=e^0=1 \therefore \lim_{x\to 0} f(x)=1$

b. Show the limits of *U(x)* and give the limit of *f(x)* in +∞ and -∞ 2pts

$$\textit{from } [2]: \lim_{x\to\pm\infty} x\ln\left(1+\frac{1}{x^2}\right)=0$$

exp is continuous, then $\lim_{X\to 0}\exp(X)=\exp(0)=e^0=1 \therefore \lim_{x\to 0} f(x)=1$

3. Complete the **chart** of f with all the previous results : 2pts
[one can admit that a≈ -0.5 ; b ≈ 0.5 ; f(-0.5) ≈ 0.4 ; f(0.5) ≈ 2.2]

x	- ∞		-1		- 0.5		0		0.5		2		+ ∞
***Sign** [f'(x)]*			—		0	⊕	‖	⊕	0		—		
***Variations and limits** of f(x)*	1	↘	0.5	↗	0.4	↗	①	↗	2.2	↘	2	↘	1

4. **Graph** of the function f *(show the asymptotes if any)* 3pts