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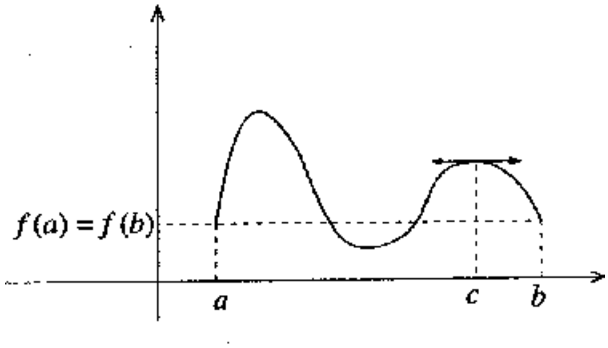
Applications of derivatives and integrals

Rolle – MVT – L'Hospital

I. Rolle's theorem *[French - XVIIth AD]*:

$$\textit{Hypothesis}\begin{cases} f \text{ is continuous on } [a;b] \\ f \text{ is differentiable on } [a;b] \\ f(a)=f(b) \end{cases}$$

$$\textit{Conclusion}\begin{cases} \text{There is at least one } c\in[a;b] \\ \text{such that } f'(c) \;=\; 0 \end{cases}$$

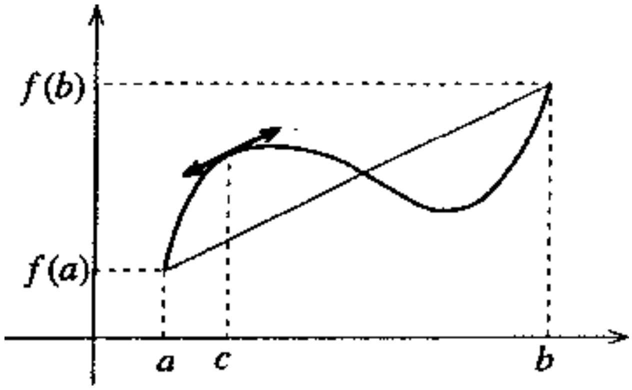


<u>Graphic interpretation</u> : if a curve joins two points without discontinuity, then obviously there must be a point of the curve corresponding to a Maximum or a Minimum of f on [a;b], and since f is differentiable the tangent line in that point is parallel to (Ox) axis.

II. The Mean Value Theorem :

$$\textit{Hyp}\;:\begin{cases} f \text{ is continuous on } [a;b] \\ f \text{ is differentiable on } [a;b] \end{cases}$$

$$\Rightarrow\begin{cases} \text{There is at least one } c\in[a;b] \\ \text{such that } f'(c) \;=\; \dfrac{f(b)-f(a)}{b-a} \end{cases}$$



<u>Graphic interpretation</u> : if a curve joins to points without discontinuity, then obviously there <u>must</u> be a point of the curve where the tangent line is parallel to the secant.

III. Equivalence between Rolle's and the Mean Value Theorem.

a. M.V.T. ⇒ Rolle's *Easy* ! If $f(b) = f(a)$ then $f'(c) = 0$. *Bingo !!!*

b. Rolle's ⇒ M.V.T. : let's consider the function g defined by the distance between the curve and the secant line between the two extreme points :
$g(x) = f(x) - s(x)$ where s(x) is the function associated to the secant line :
for any point (x,y) on the secant line we have :

$$\frac{y-f(a)}{x-a}=\frac{f(b)-f(a)}{b-a} \;\Leftrightarrow\; y=s(x)=(x-a)\frac{f(b)-f(a)}{b-a}+f(a)$$

$$\text{and then } g(x) \;=\; f(x)-\left[\;(x-a)\frac{f(b)-f(a)}{b-a}+f(a)\right]$$

This new function g is indeed continuous & differentiable on [a;b] since f and s are so.

we also have g(a)=0 and g(b)=0 hence g(a)=g(b), and $g'(x) \;=\; f'(x)-\dfrac{f(b)-f(a)}{b-a}$

Therefore all hypotheses of Rolle's theorem apply to the function g :

<u>Conclusion</u> :there is one c∈[a;b] such that g'(c) = 0 ⇔ $f'(c) \;=\; \dfrac{f(b)-f(a)}{b-a}$. Finished!

<u>Note</u> : we have not provided here the proof of either ones of these two theorems : too advanced !!! a formal proof of the MVT can be found in some higher degree calculus books. The equivalence of the two theorems does not give a proof to either ones !

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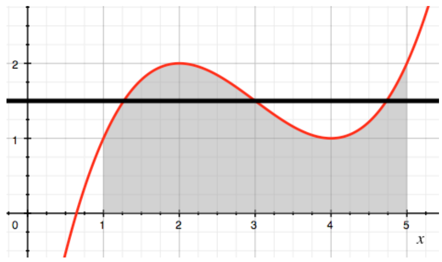
IV. Mean value of a function on an interval [a;b]

Let $F(x)=\int_a^x f(t).dt$ *then we know that if is continuous on [a;b) F is continuous and differentiable on [a;b] and F'(x)=f(x).*

We also know that $F(b)-F(a)=\int_a^b f(t).dt$ *then from MVT there is a c∈[a;b] such that*

$$F'(c)=\frac{F(b)-F(a)}{b-a}\Leftrightarrow f(c)=\frac{1}{b-a}\int_a^b f(t).dt$$

$$\Leftrightarrow \int_a^b f(t).dt=(b-a)f(c)$$



By definition this number $\mu = f(c)$ is called the **mean value of the function f on [*a*;*b*]**.

μ represents the **height of a rectangle of base [a;b] having the same area** that the domain defined by the curve of *f* and the [0*x*] axis on [*a*;*b*].

Ex $f(x)=\frac{1}{4}x^3-\frac{9}{4}x^2+6x-3$; $\int_1^5 f(x)dx=\left[\frac{x^4}{16}-3\frac{x^3}{4}+3x^2-3x\right]_1^5=6\ \therefore \mu=\frac{1}{5-1}\int_1^5 f(x)dx=\frac{6}{4}=1.5\quad (Cf.picture)$

Algebraic interpretation μ = limit of the arithmetic mean of n values of f on [a;b] :

$$\int_a^b f(t).dt=\lim_{n\to\infty}\sum_{i=1}^n f(x_i)\frac{b-a}{n}=(b-a)\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}f(x_i)=(b-a)\lim_{n\to\infty}\frac{f(x_1)+f(x_2)+...+f(x_n)}{n}$$

Therefore $$\boxed{\lim_{n\to\infty}\frac{f(x_1)+f(x_2)+...+f(x_n)}{n}=\frac{1}{b-a}\int_a^b f(t).dt=\mu}$$

- It is indeed this formula which justifies the definition of "***mean value of f on [a;b]***"*and of the MVT.*

Important remark *[Thank you 孟庆晗 !!!] :In some books the proof of the formula F'(x) =f(x) is given by using the mean value theorem applied to the function F. Hence we would think that the above formula of the mean value of f on [a;b] seems to be using itself to prove itself, which would not be a correct way of reasonning. But it's possible to prove that F'(x) =f(x) without using the mean value of f. So we are safe ! In fact when the mean value of is used to prove that F'(x)=f(x), it is implicitely using the MVT, then the above result is just a retranscription of the MVT.*

V. Extended formula of the MVT :

If f and g are differentiable and $g'(x)\neq 0$ *on* [*a* ;*b*], *there is* $c\in[a;b]$ *such that* $\frac{f'(c)}{g'(c)}=\frac{f(b)-f(a)}{g(b)-g(a)}$

Proof : *h(x)= f(x)[g(b)-g(a)]-g(x)[f(b)-f(a)]-f(a)g(b)+f(b)g(a) matches Rolle's conditions: h(a) = 0 and h(b) =0 and h'(x) = f'(x)[g(b)-g(a)]-g'(x)[f(b)-f(a)] then there is c∈[a;b] such that h'(c) = 0 ⇔ f'(c)[g(b)-g(a)] = g'(c)[f(b)-f(a)]. Done !*

VI. L'Hospital's rule *[French - XVIIth AD]* to solve indeterminate forms $\frac{0}{0}$ or $\frac{\infty}{\infty}$

$$\boxed{\left\{\lim_{x\to a}f(x)=0,\lim_{x\to a}g(x)=0,\ for\ x\neq a,\ g(x)\neq 0,g'(x)\neq 0\right\}\Rightarrow \lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}}$$

Proof : *On [a ; x] f and g are differentiable and g ≠ 0 then from the extended MVT :*

there is c(x) $\in[a;x]$ *such that* $\frac{f'(c(x))}{g'(c(x))}=\frac{f(x)-f(a)}{g(x)-g(a)}$. *To make f and g continuous at a we fix f(a) = 0*

and g(a)=0, then when x → a ; c(x) → a and. $\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(c(x))}{g'(c(x))}=\lim_{x\to a}\frac{f'(x)}{g'(x)}$.

Note *if f 'and g' are continuous in a and g'(a)≠0 then this rule comes directly from the following :*

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f(x)-f(a)}{x-a}\frac{x-a}{g(x)-g(a)}=\lim_{x\to a}\frac{f(x)-f(a)}{x-a}\lim_{x\to a}\frac{x-a}{g(x)-g(a)}=f'(a)\frac{1}{g'(a)}=\frac{f'(a)}{g'(a)}$$

- *L'hospital's rule is most powerful when f '(a) and g'(a) are not both defined or if g'(a) =0 [See Exercises]*
- *The rule holds for* $\frac{\infty}{\infty}$ and when a = ∞ $\left\{\lim_{x\to a}f(x)=\infty,\lim_{x\to a}g(x)=\infty,\ g\neq 0,g'\neq 0\right\}\Rightarrow \lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}$ *(admitted).*