# Model for a complete study of a rational function

Reminders : 1°) The **set of definition** of the function must be defined in terms of **Intervals**.

2°) The **limits** of a rational function when the variable goes to infinity is the same as the ratio of its higher degrees terms.

3°) The horizontal and vertical **asymptotes** are determined by the limits of *f* at on the boundaries of each interval of the set of definition.

4°) The **variations** of a function are determined by the **sign of it's derivative**. Therefore it's, most of the time, necessary to calculate the derivative, determine it's zeroes and study whether or not there is a change of sign. If there is no change of sign then it's not a **maximum** or **minimum**, it's an **inflexion** point.

5°) A line (D) of equation $y = ax + b$ is an **oblique asymptote** to the curve representing the graph of the function if and only if the **difference *[f(x) – (ax+b)]*** tends towards 0 when $x$ goes to infinity. The **position of the curve** with respect of the line (D) must be determined by the **sign** of that difference.

6°) The **zeroes** of a function can be approximately determined by observing the **changes of sign** in a given interval : if *f* is a derivable (hence continuous) and monotonous function on [a ; b] and if $f(a).f(b) < 0$ then there is one and only one α in [a ; b] such that $f(\alpha) = 0$. Then if the interval [a ; b] is small, a and b can be considered as approximate values of α.

7°) Summarize all the previous studies in a **chart** showing the boundaries of the definition set, the zeroes of the function, the zeroes of its derivative, the values of maximum or minimum, and the limits.

8°) Eventually study the **symmetries** with respect to a given vertical axis or a given point.

9°) Find the equation of **tangent** line at the intersection of the curve of the function with the "vertical" axis. Study the case of singular points (vertical tangent, half tangent, inflexion point).

10°)Draw carefully the curve after having carefully placed the asymptotes, the zeroes, the maximum and minimum, and the intersections with the axis.

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$$f(x) = \frac{2x^3 - x^2 + 2}{(2x-3)^2}$$

1. *Set of definition :* $D_f = \mathbb{R} \setminus \{\frac{3}{2}\} = ]-\infty ; \frac{3}{2}[ \bigcup ]\frac{3}{2} ; +\infty[$

2. $\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty}\left(\frac{2x^3}{4x^2}\right) = \lim_{x\to+\infty}\frac{x}{2} = +\infty$ ; $\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty}\left(\frac{2x^3}{4x^2}\right) = \lim_{x\to-\infty}\frac{x}{2} = -\infty$ ; $\lim_{x\to\frac{3}{2}} f(x) = +\infty$

*Therefore there are 4 infinite branches, with one vertical asymptote x=3/2. To find the oblique asymptotes one can use two methods :*

*a) Break the given expression of f(x) into fractions such as* $f(x) = ax + b + \frac{cx+d}{(2x-3)^2}$ *and find a,b,c,d, by identification or by Euclidian division of the polynomials.*

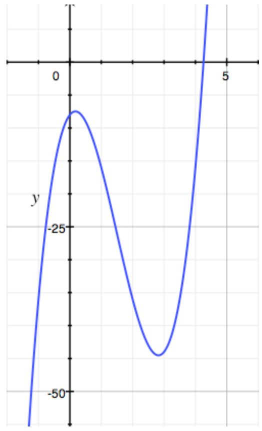
*b) Calculate* $\lim_{x\to\infty}\frac{f(x)}{x} = a$*, then* $\lim_{x\to\infty}[f(x) - ax] = b$*. But in both cases we need to know the sign of the difference [f(x)-(ax +b)] to be able to fix the position of the curve with respect to the asymptote (D) defined by y=ax+b. In this case we find* $f(x) = \frac{1}{2}x + \frac{5}{4} + \frac{\frac{21}{2}x - \frac{37}{4}}{(2x-3)^2}$ *=> asymptote equation* $y = \frac{1}{2}x + \frac{5}{4}$

*because* $\lim_{x\to+\infty}\left[f(x) - \left(\frac{1}{2}x + \frac{5}{4}\right)\right] = \lim_{x\to+\infty}\frac{\frac{21}{2}x - \frac{37}{4}}{(2x-3)^2} = \lim_{x\to+\infty}\left[\frac{21x}{8x^2}\right] = \lim_{x\to+\infty}\frac{21}{8x} = 0^+ (\text{or } 0^- \text{ for } -\infty)$

*Therefore (Cf) is above (D) for* $x > \frac{37}{42} \simeq 0.8$ *and under (D) if* $x < \frac{37}{42} \simeq 0.8$ *(the change of sign corresponds to the point where the curve crosses it's asymptote).*

3. *Derivative* $f'(x) = \dfrac{4x^3 - 18x^2 + 6x - 8}{(2x-3)^3}$

4. *The Zeroes of the derivative are the Zeroes of* $P(x) = 4x^3 - 18x^2 + 6x - 8$ .
*Therefore we must study the variations of P to examine its zeroes and sign. To do so we must study the sign of* $P'(x) = 6(2x^2 - 6x + 1)$ *and chart the variations of P*
*(The zeroes of P'(x) are* $\frac{3-\sqrt{7}}{2} \simeq 0.2$ *and* $\frac{3+\sqrt{7}}{2} \simeq 2.8$*)*




| *x* | - ∞ | | 0.2 | | 2.8 | | α | +∞ |
|---|---|---|---|---|---|---|---|---|
| *Sign of [P '(x)]* | | + | 0 | - | **0** | + | | |
| *SIGN of P* | | - | -7.4 | - | -44.5 | **-** | **0** | + |

*From the graph we can see that* α *is approximately equal to 4.2, and we can check by calculations that P(4.2) <0 and P(4,3) >0 hence* α ≈ 4.25

*The sign of f'(x) is the same as the sign of P(x) for x > 1.5 and opposite for x < 1.5*

| *x* | - ∞ | -0.9 | | 0 | | 1.5 | | 4.25 | | + ∞ |
|---|---|---|---|---|---|---|---|---|---|---|
| *Sign [f'(x)]* | | | + | | | \|\| | - | 0 | + | |
| Variations *and* ***limits*** of *f* | - ∞ | 0 | ↗ | 0.2 | ↗ | + ∞ \|\| + ∞ | ↘ | min = 4.5 | ↗ | + ∞ |