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*Applications of derivatives and integrals*

# Finite Variations Inequalities

*and their applications on sequences*

## 1. Finite Variations Inequalities :

1. *General form : If a function f is differentiable on* $[a;b]$ *and such that for any* $x \in [a;b]$, $m \le f'(x) \le M$, *then for any interval* $[u;v] \subseteq [a;b]$ $m \le \dfrac{f(v)-f(u)}{v-u} \le M$
2. *Particular form* : $m(b-a) \le f(b) - f(a) \le M(b-a)$
3. *Reduced form* : $|f'(x)| \le M \Rightarrow |f(b) - f(a)| \le M|b-a|$

***Proof*** *: from the MVT we know that there is one* $c \in [u;v]$ such that $f'(c) = \dfrac{f(v)-f(u)}{v-u}$

*Then the initial conditions on f '(x) give the result.*

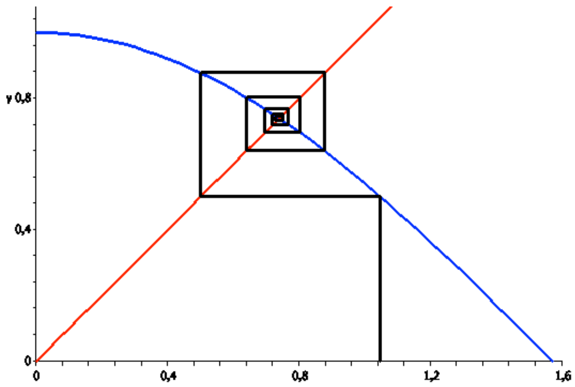
## 2. Application to prove the convergence of some recursive sequences :

1. *General case :Let* $(u_n)$ *be defined by* $u_{n+1} = f(u_n)$ *and* $u_0$ *fixed, with f defined on* $[a;b]$*, such that :*
   i. *for any* $n \in N$, $a \le u_n \le b$
   ii. *There is a fixed point* $\alpha = f(\alpha)$, $\alpha \in [a;b]$
   iii. $|f'(x)| \le k$ *for any x on* $[a;b]$ *with* $0 < k < 1$*. Then for* $n \in N$,
   $|u_{n+1} - \alpha| = |f(u_n) - f(\alpha)| \le k.|u_n - \alpha|$

*Therefore, by immediate recurrence we have ,*

$$|u_n - \alpha| \le k^n.|u_0 - \alpha|$$

*Since* $\lim k^n = 0$ *, then* $\lim|u_n - \alpha| = 0$ *which means that* $\lim u_n = \alpha$ *(fixed point of f).*




2. *Example : f = cosine, a=0, b=π/3,* $u_0 = \pi/3$, $f'(x) = -\sin(x) \Rightarrow |f'(x)| \le \sqrt{3}/2$

*there is obviously a fixed point* $\alpha$ *, such that* $\alpha = \cos(\alpha)$ *(Inerception of the first bisector (y=x) with the curve of Cosine on the interval* $[0 ; \pi/3]$*.*

*Then we have* $|u_n - \alpha| \le k^n.|u_0 - \alpha|$ *with* $k = \sqrt{3}/2 \therefore |k| < 1$.

*Hence we may obtain an approximate value of* $\alpha$ *by majoring* $.|u_0 - \alpha| = |\pi/3 - \alpha|$ *by* $\pi/6$ *and then for n =10 we would have* $|u_{10} - \alpha| \le (\sqrt{3}/2)^{10}.\pi/6 = 0.12$

*which means that* $\alpha = \cos(\cos(\cos(\cos(\cos(\cos(\cos(\cos(\cos(\cos(\pi/3))))))))))$ *with an error less than* $10^{-2}$*. A calculator gives* $\alpha \approx 0.75$ *radians or approximately 43°.*