jiguanglaoshi@gmail.com - http://beijingshanmaths.org

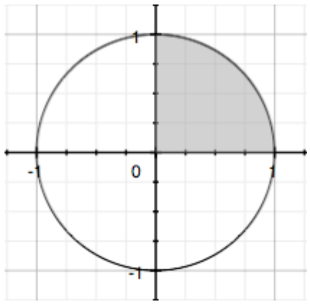
# ∫ Extra change of variables ∫

**I. Use an adequate change of variable to calculate the following integrals :**

1. $I = \int_0^1 \sqrt{1-x^2}\,dx$ *let* $x = \sin u \Leftrightarrow u = \arcsin x$ *and* $dx = \cos u\, du$

$x = 0 \Rightarrow u = \arcsin(0) = 0$ *and* $x = 1 \Rightarrow u = \arcsin(1) = \frac{\pi}{2}$ *then* :

$I = \int_0^{\frac{\pi}{2}} (\cos u)^2 du = \int_0^{\frac{\pi}{2}} \frac{\cos 2u + 1}{2}\, du = \frac{1}{2}\left[\frac{\sin 2u}{2}\right]_0^{\frac{\pi}{2}} + \frac{1}{2}[u]_0^{\frac{\pi}{2}} = \frac{\pi}{4}$




2. $J = \int_0^1 \frac{1}{\sqrt{1-x^2}}\,dx$ *let* $x = \sin u \Leftrightarrow u = \arcsin x$ *and* $dx = \cos u\, du$

$x = 0 \Rightarrow u = \arcsin(0) = 0$ *and* $x = 1 \Rightarrow u = \arcsin(1) = \frac{\pi}{2}$ *then* :

$I = \int_0^{\frac{\pi}{2}} \frac{\cos u}{\sqrt{1-\sin^2 u}}\, du = \int_0^{\frac{\pi}{2}} \frac{\cos u}{\cos u}\, du = \int_0^{\frac{\pi}{2}} 1\, du = [u]_0^{\frac{\pi}{2}} = \frac{\pi}{2}$

3. $K = \int_0^1 \frac{1}{1+x^2}\,dx$ *let* $x = \tan u \Rightarrow dx = (1+\tan^2 u).du$ ; $u = \arctan x$

$x = 0 \Rightarrow u = \text{arcs}\tan(0) = 0$ *and* $x = 1 \Rightarrow u = \text{arcs}\tan(1) = \frac{\pi}{2}$ *then* :

$K = \int_0^{\frac{\pi}{2}} \frac{1+\tan^2 u}{1+\tan^2 u}\, du = \int_0^{\frac{\pi}{2}} 1\, du = [u]_0^{\frac{\pi}{2}} = \frac{\pi}{2}$

**II. Without calculating the value of each integral, prove by an adequate change of variable that :**

1. $I = \int_0^{\frac{\pi}{2}} (\sin x)^3 \cos x\, dx$ and $J = \int_0^{\frac{\pi}{2}} (\cos x)^3 \sin x\, dx$, prove that I = J

$I = \int_0^{\frac{\pi}{2}} (\sin x)^3 \cos x\, dx$ *let* $x = \frac{\pi}{2} - u \Rightarrow dx = -du$ ; $u = \frac{\pi}{2} - x$

$\sin x = \sin\left(\frac{\pi}{2} - u\right) = \cos u$ *and* $\cos x = \cos\left(\frac{\pi}{2} - u\right) = \sin u$

$x = 0 \Rightarrow u = \frac{\pi}{2}$ *and* $x = \frac{\pi}{2} \Rightarrow u = 0$ *then*

$I = \int_{\frac{\pi}{2}}^0 (\cos u)^3 \sin u\, (-du) = -\int_{\frac{\pi}{2}}^0 (\cos u)^3 \sin u\, du = \int_0^{\frac{\pi}{2}} (\cos u)^3 \sin u\, du = J$

2. $K_n = \int_{-1}^1 (1-x^2)^n\, dx$ and $K'_n = \int_{-1}^1 (x^2-1)^n\, dx$, prove that $K'_n = (-1)^n K_n$

$K'_n = \int_{-1}^1 (1-x^2)^n\, dx = (-1)^n \int_{-1}^1 (x^2-1)^n\, dx$ No Change of variable !!!

3. $I_n = \int_0^{\frac{\pi}{2}} (\sin x)^n\, dx$ and $J_n = \int_0^{\frac{\pi}{2}} (\cos x)^n\, dx$, prove that $I_n = J_n$

$$I_n = \int_0^{\frac{\pi}{2}} (\sin x)^n\, dx \quad let\ x = \frac{\pi}{2} - u \Rightarrow dx = -du\ ;\ u = \frac{\pi}{2} - x$$

$$\sin x = \sin\left(\frac{\pi}{2} - u\right) = \cos u\ ;\ x = 0 \Leftrightarrow u = \frac{\pi}{2}\ ;\ x = \frac{\pi}{2} \Leftrightarrow u = 0$$

$$\therefore\ I = \int_{\frac{\pi}{2}}^{0} (\cos u)^n (-du) = -\int_{\frac{\pi}{2}}^{0} (\cos u)^n du = \int_0^{\frac{\pi}{2}} (\cos u)^n du = J_n$$

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4. $K_n = \int_{-1}^{1} (1-x^2)^n\, dx$ and $I_n = \int_0^{\frac{\pi}{2}} (\sin x)^n\, dx$, prove that $K_n = 2\, I_{2n+1}$

$$K_n = \int_{-1}^{1} (1-x^2)^n\, dx \quad let\ x = \sin u \Rightarrow dx = \cos u . du\ ;$$

$$1 - x^2 = \cos^2 u\ ;\ x = -1 \Leftrightarrow u = -\frac{\pi}{2}\ ;\ x = 1 \Leftrightarrow u = \frac{\pi}{2}$$

$$\therefore\ K_n = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos u)^{2n+1} du = \int_{-\frac{\pi}{2}}^{0} (\cos u)^{2n+1} du + \int_0^{\frac{\pi}{2}} (\cos u)^{2n+1} du = 2J_{2n+1} = 2I_{2n+1}$$

$$\because\ \int_{-\frac{\pi}{2}}^{0} (\cos u)^{2n+1} du = \int_{\frac{\pi}{2}}^{0} (\cos(-u))^{2n+1} d(-u) = -\int_{\frac{\pi}{2}}^{0} (\cos u)^{2n+1} du = \int_0^{\frac{\pi}{2}} (\cos u)^{2n+1} du$$

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5. Calculate $F(x) = \int_x^1 \ln\left(1 + \frac{1}{t}\right) dt \quad and \quad G(x) = \int_{-1}^{x} \ln\left(1 - \frac{1}{t}\right) dt$, prove that $G(x) = F(-x)$

$$G(x) = \int_{-1}^{x} \ln\left(1 - \frac{1}{t}\right) dt \quad let\ u = -t \Rightarrow dt = -du\ ;\ t = -1 \Leftrightarrow u = 1\ ;\ t = x \Leftrightarrow u = -x$$

$$\therefore \int_{-1}^{x} \ln\left(1 - \frac{1}{t}\right) dt = \int_1^{-x} \ln\left(1 + \frac{1}{u}\right)(-du) = -\int_1^{-x} \ln\left(1 + \frac{1}{u}\right) du = \int_{-x}^{1} \ln\left(1 + \frac{1}{u}\right) du = F(-x)$$

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6. Calculate $F(x) = \int_0^x t^2 e^t dt \quad and \quad G(x) = \int_0^x t^2 e^{-t}\, dt$, prove that $G(x) = -F(-x)$

$$G(x) = \int_0^x t^2 e^{-t}\, dt \quad let\ u = -t \Rightarrow dt = -du\ ;\ t = 0 \Leftrightarrow u = 0\ ;\ t = x \Leftrightarrow u = -x$$

$$\therefore\ \int_0^x t^2 e^{-t}\, dt = \int_0^{-x} (-u)^2 e^u\, d(-u) = -\int_0^{-x} u^2 e^u\, du = -F(-x)$$

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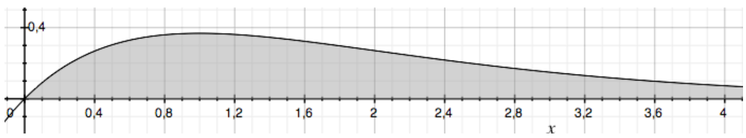
**III.**

$$Let\ F(x) = \int_0^x t e^{-pt}\, dt = \left[-\frac{1}{p} t e^{-pt} dt\right]_0^x + \frac{1}{p}\int_0^x e^{-pt}\, dt$$

$$= \left[-\frac{1}{p} t e^{-pt} dt - \frac{1}{p^2} e^{-pt}\right]_0^x = -\frac{1}{p} x e^{-px} dt - \frac{1}{p^2} e^{-px} + 0 + \frac{1}{p^2}$$

$$and\ \lim_{x\to+\infty} F(x) = -0 - 0 + 0 + \frac{1}{p^2} \Leftrightarrow \int_0^{+\infty} t e^{-pt}\, dt = \frac{1}{p^2}$$




$$F_n(x) = \int_0^x t^n e^{-pt}\, dt = \left[-\frac{1}{p} t^n e^{-pt}\right]_0^x + \frac{n}{p} F_{n-1}(x) = \left[-\frac{e^{-pt}}{p}(t^n + t^{n-1} + \ldots + t\right]_0^x + \frac{n!}{p^n} F_0(x)$$

$$\therefore \int_0^{+\infty} t^n e^{-pt}\, dt = \lim_{x\to+\infty} \int_0^x t^n e^{-pt}\, dt = \frac{n!}{p^n}\int_0^{+\infty} e^{-pt}\, dt = \frac{n!}{p^n}\frac{1}{p}\lim_{x\to+\infty}\left[-e^{-ptx} + 1\right]_0^x = \frac{n!}{p^{n+1}}$$