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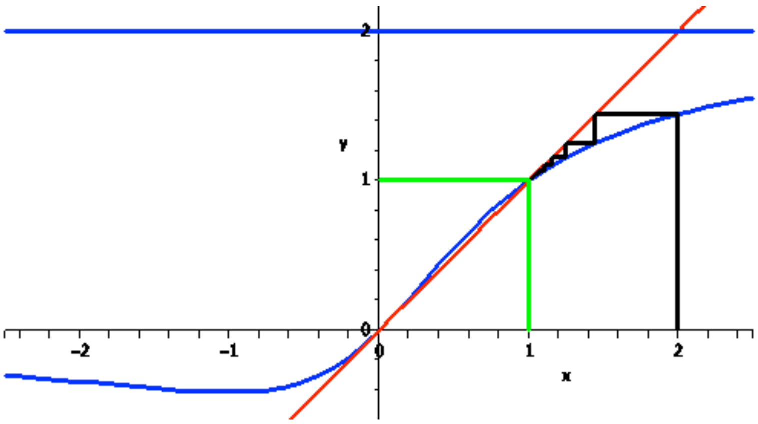
Finite Variations Inequalities

and their applications on sequences

Let f be the function defined by : $f(x)=1+\dfrac{x-1}{\sqrt{x^2+1}}$ *and* $u_{n+1}=f(u_n)$ *with* $u_0=2$.

1°) *Study the limits of f(x) on]-∞ ; +∞[and give the equations of the asymptotes.*
2°) *Calculate f'(x) and study it's sign and give the variations of f.*
3°) *Show that there is at least on fixed point for f, such that f(x) =x.*
4°) *Draw the graph of f on [-2 ; 2] and show the construction of the first terms of* (u_n)
5°) *Research of a majorant M, 0 < M < 1 for |f'(x) | on [1;2] :*
   i. *Calculate f''(x), second derivative of f, on on [1;2]*
   ii. *Study the Sign of f''(x) and chart the variations of f'(x) on [1;2].*
   iii. *Show that for x ∈[1;2], |f(x)| ≤ 1/√2*
6°) *Use the Finite Variations Inequalities to prove that* $\lim u_n=1$.



Limits : $\begin{cases}\lim\limits_{x\to+\infty} f(x)=2\\ \lim\limits_{x\to-\infty} f(x)=0\end{cases}$

Asymptotes : $\begin{cases}y=2 \text{ for } x\to+\infty\\ y=0 \text{ for } x\to-\infty\end{cases}$

First derivative : $f'(x)=\dfrac{x+1}{\left(x^2+1\right)^{\frac{3}{2}}}$

Variations of f : f'(x) ≥ 0 ⇔ x ≥ -1

∴ *f is increasing on [-1 ;+∞[and decreasing on]-∞ ; -1]*

Second derivative : $f''(x)=-\dfrac{2x^2+3x-1}{\left(x^2+1\right)^{\frac{5}{2}}}$

Variations of f'(x) : $f''(x)\ \geq\ 0 \iff \dfrac{-3-\sqrt{17}}{4}\leq x\leq\dfrac{-3+\sqrt{17}}{4}\simeq 0.3<1$

Then f'(x) is decreasing on [1;2] ⇒ f(2) ≤ f'x) ≤ f'(1) . f'(2) ≈ 0.2 and f'(1)=1/ √2 ≈ 0.7.

Hence on [1;2] |f'(x)| ≤ 1/ √2 Then by the FVI, we get :

For n ∈N, $|u_{n+1}-1|=|f(u_n)-f(1)|\leq k.|u_n-1|\Rightarrow |u_n-1|\leq k^n.|u_0-1|=\left(\dfrac{1}{\sqrt{2}}\right)^n$

$$\lim_{n\to\infty}\left(\frac{1}{\sqrt{2}}\right)^n=0\Rightarrow\lim_{n\to\infty}|u_n-1|=0\Rightarrow\lim_{n\to\infty}u_n=1$$