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Problem II.(6.1) : Let $F(x) = \left(1+\frac{1}{x}\right)^{x^2}$

1. By definition the function F is composed of the two functions $f(x) = \left(1+\frac{1}{x}\right)$ and $u(x) = x^2$

   such that $F(x) = Exp\left[\ln[(f(x))^{u(x)}]\right] = Exp(u(x)\ln[f(x)]) = e^{u(x)\ln[f(x)]} = e^{x^2\ln\left(1+\frac{1}{x}\right)}$

   Therefore *f(x)* must be strictly positive, which means *x < - 1 or x > 0*

2. The derivative of F(x) is $F'(x) = [f(x)]^{u(x)}\left[u'(x).\ln[f(x)] + u(x).\frac{f'(x)}{f(x)}\right]$ $\because$ $ln'[f(x)] = \frac{f'(x)}{f(x)}$

$$F'(x) = \left(1+\frac{1}{x}\right)^{x^2}\left[2x.\ln\left(1+\frac{1}{x}\right) + x^2.\frac{-\frac{1}{x^2}}{\frac{x+1}{x}}\right] = x\left(1+\frac{1}{x}\right)^{x^2}\left[2\ln\left(1+\frac{1}{x}\right) - \frac{1}{x+1}\right]$$

3. $Sgn[F'(x)] = Sgn\ x\left[2\ln\left(1+\frac{1}{x}\right) - \frac{1}{x+1}\right]$ *for* $x < -1$ *or* $x > 0$. *Let* $u(x) = 2\ln\left(1+\frac{1}{x}\right) - \frac{1}{x+1}$

$u'(x) = 2\frac{-\frac{1}{x^2}}{\frac{x+1}{x}} + \frac{1}{(x+1)^2} = -\frac{x+2}{x(x+1)^2}$ Hence u'(x) > 0 for -2 < x < -1 and u'(x) < 0 for x > 0 or x < -2

4. **Study of the Variations of u and F** :
The function u is decreasing on]-∞ ; -2 [∪]0 ; +∞[and increasing on]-2 ; 0 [, with a minimum m = u(-2) = -0.4 < 0. $\lim_{x\to\pm\infty} u(x) = 0$ $\because$ $\ln(1) = 0$

then for $x < -1$,

$$u(x) = \frac{-1}{x+1}(1 + (-x-1)\ln(-x-1) + (x+1)\ln(-x))$$

$$\lim_{x\to -1^-} u(x) = \lim_{x\to -1^-} \frac{-1}{x+1}(1+0+0) = +\infty$$

x	$-\infty$ -2 α -1 0 $+\infty$
Sign [u'(x)]	— 0 ⊕ ||//////////|| —
Variations & Sign of u(x)	0^- ↘ **m** ↗ $+\infty$||//////////|| ↘ — 0 ⊕ ||//////////|| ⊕ 0^+
Sgn F'(x)	⊕ 0 — ||//////////|| ⊕ 0^+
Var. of F(x)	0^+ ↗ M ↘ 0^+||//////////|| 1 ↗ $+\infty$

Sgn[F'(x)]=Sgn[x.u(x)], u(x) changes of sign at x=α, (-2<α<-1), u(α)=0 ∴ F'(α)=0 ∴ M=F(α)=0.1 Max.

5. **Study of the limits of F at the ends of the intervals]-∞ ; -1 [∪] 0 ; +∞ [**

(a) $\lim_{x\to -1^-} F(x) = 0^+$ because $\lim_{x\to -1^-} x^2\ln\left(1+\frac{1}{x}\right) = -\infty$, and $\lim_{X\to -\infty}[ExpX] = 0^+$

(b) $\lim_{x\to 0^+} F(x) = 1^+$ because $\lim_{x\to 0^+} x^2\ln\left(1+\frac{1}{x}\right) = \lim_{x\to 0^+}[x\ln(1+x)] + \lim_{x\to 0^+}[(x\ln x] = 0\times\ln 1 + 0 = 0$

($\lim_{X\to 0+}[X\ln X] = 0^-$) , then by continuity of Exp, $\lim_{X\to 0}[ExpX] = e^0 = 1$ ∴ $\lim_{x\to 0^+} F(x) = 1$

Hence to **extend the function F by continuity** at x = 0 and x = -1 we may fix F(0) = 1 and F(-1) = 0.

(c) $\lim_{x\to -\infty} F(x) = 0^+$ because $\lim_{x\to -\infty} x^2\ln\left(1+\frac{1}{x}\right) = \lim_{x\to -\infty} x\left[x\ln\left(1+\frac{1}{x}\right)\right] = -\infty\times 1 = -\infty$

$\because$ $\lim_{x\to -\infty} x\ln\left(1+\frac{1}{x}\right) = 1$ then $\lim_{X\to -\infty}[ExpX] = 0^+$ ∴ $\lim_{x\to -\infty} F(x) = 0^+$

(d) $\lim_{x\to +\infty} F(x) = +\infty$ because $\lim_{x\to +\infty} x^2\ln\left(1+\frac{1}{x}\right) = \lim_{x\to -\infty} x\left[x\ln\left(1+\frac{1}{x}\right)\right] = +\infty\times 1 = +\infty$

Problem II.(6.2) : Let $F(x)=\left(-1-\dfrac{1}{x}\right)^{x^2}$

1. By definition the function F is composed of the two functions $f(x)=\left(-1-\dfrac{1}{x}\right)$ and $u(x)=x^2$

   such that $F(x)=Exp\left[\ln[(f(x))^{u(x)}\right]=Exp\left(u(x)\ln[f(x)]\right)=e^{u(x)\ln[f(x)]}=e^{x^2\ln\left(-1-\frac{1}{x}\right)}$

   Therefore $f(x)$ must be strictly positive, which means $-1<x<0$

2. The derivative of F(x) is $F'(x)=[f(x)]^{u(x)}\left[u'(x).\ln[f(x)]+u(x).\dfrac{f'(x)}{f(x)}\right]$ $\because$ $ln'[f(x)]=\dfrac{f'(x)}{f(x)}$

$$F'(x)=\left(-1-\frac{1}{x}\right)^{x^2}\left[2x.\ln\left(-1-\frac{1}{x}\right)+x^2.\frac{\frac{1}{x^2}}{-\frac{x+1}{x}}\right]=x\left(-1-\frac{1}{x}\right)^{x^2}\left[2\ln\left(-1-\frac{1}{x}\right)-\frac{1}{x+1}\right] \text{ for } -1<x<0$$

3. $Sgn[F'(x)]=-Sgn\left[2\ln\left(-1-\dfrac{1}{x}\right)-\dfrac{1}{x+1}\right]$ *for* $-1<x<0$. *Let* $u(x)=2\ln\left(-1-\dfrac{1}{x}\right)-\dfrac{1}{x+1}$

$$u'(x)=2\frac{\frac{1}{x^2}}{-\frac{x+1}{x}}+\frac{1}{(x+1)^2}=-\frac{x+2}{x(x+1)^2}$$ Hence u'(x) > 0 for - 1 < x < 0

4. **Study of the Variations of u and F** :

   The function u is increasing on]-1 ; 0[

$$u(x)=\frac{-1}{x+1}[1-(x+1)\ln(x+1)+(x+1)\ln(-x)]$$

$$\lim_{x\to-1^+}u(x)=\lim_{x\to-1^+}\frac{-1}{x+1}(1+0+0\times0)=-\infty$$

$$\lim_{x\to0^-}u(x)=\lim_{x\to0^-}2\ln\left(-1-\frac{1}{x}\right)-\frac{1}{x+1}=+\infty$$

x	- ∞	-1		α		0	+ ∞
Sign [u'(x)]	//////////	‖		⊕		‖	//////////////////
Variations &	//////////	‖ -∞	↗	0	↗	+∞‖	//////////////////
Sign of u(x)	//////////	‖	—	0	⊕	‖	//////////////////
Sgn F'(x)	//////////	‖	⊕	0	—	‖	//////////////////
Var. of F(x)	//////////	‖ 0	↗	M	↘	1‖	//////////////////

   The function u being continuous and strictly monotonous on]-1 ; 0[, u(x) changes of sign in one point only at x= α =-.3, u(α)=0

   Sgn[F'(x)] = -Sgn[u(x)] ∴ F'(α) = 0 ∴ M=F(α) = 1.1 is a Maximum. See the above Chart.

5. **Study of the limits of F at the ends of the interval]-1 : 0 [**

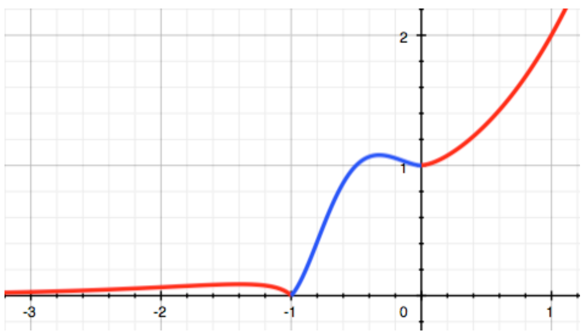
(a) $\lim_{x\to-1^+}F(x)=0^+$ because $\lim_{x\to-1^+}x^2\ln\left(-1-\dfrac{1}{x}\right)=-\infty$, and $\lim_{X\to-\infty}[ExpX]=0^+$

(b) $\lim_{x\to0^-}F(x)=1^-$ because $\lim_{x\to0^-}x^2\ln\left(\dfrac{x+1}{-x}\right)=\lim_{x\to0^-}[x^2\ln(1+x)]+\lim_{x\to0^-}[(-x)(-x)\ln(-x)]=0\times\ln1+0=0$

($\lim_{X\to0^+}[X\ln X]=0^-$) , then by continuity of Exp, $\lim_{X\to0}[ExpX]=e^0=1$ ∴ $\lim_{x\to0^-}F(x)=1$

Hence to **extend the function F by continuity** at x = 0 and x = -1 we may fix F(0) = 1 and F(-1) = 0.

6. **Graph of the function** $F(x)=\left|1+\dfrac{1}{x}\right|^{x^2}$



It's the réunion of the graphs or the two previous functions. The approximate values of the two maximum have been found with a calculator.

The position of the tangent lines at (0;1) and at (-1;0) is a more complicated question that will be studied later.