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Problem II.(5.1) : Let $F(x) = \left(1 - \frac{1}{x}\right)^{2x}$

1. By definition the function F is composed of the two functions $f(x) = \left(1 - \frac{1}{x}\right)$ and $u(x) = 2x$

   such that $F(x) = Exp\left[\ln[(f(x))^{u(x)}\right] = Exp(u(x)\ln[f(x)]) = e^{u(x)\ln[f(x)]} = e^{2x\ln\left(1-\frac{1}{x}\right)}$

   Therefore *f(x)* must be strictly positive, which means that $x < 0$ *or* $x > 1$

2. The derivative of F(x) is $F'(x) = [f(x)]^{u(x)}\left[u'(x).\ln[f(x)] + u(x).\frac{f'(x)}{f(x)}\right]$ $\because\ ln'[f(x)] = \frac{f'(x)}{f(x)}$

$$F'(x) = \left(1-\frac{1}{x}\right)^{2x}\left[2.\ln\left(1-\frac{1}{x}\right) + 2x.\frac{\frac{1}{x^2}}{\frac{x-1}{x}}\right] = 2\left(1-\frac{1}{x}\right)^{2x}\left[\ln\left(1-\frac{1}{x}\right) + \frac{1}{x-1}\right]$$

3. $Sgn[F'(x)] = Sgn\left[\ln\left(1-\frac{1}{x}\right) + \frac{1}{x-1}\right]$ for x < 0 or x > 1 . Let u(x) = $\ln\left(1-\frac{1}{x}\right) + \frac{1}{x-1}$

$$u'(x) = \frac{\frac{1}{x^2}}{\frac{x-1}{x}} - \frac{1}{(x-1)^2} = -\frac{1}{x(x-1)^2}$$

Hence u'(x) > 0 for x <0 and u'(x) < 0 for x > 1

4. **Study of the Variations** of F : The function u is increasing on]-∞ ; 0 [and decreasing on] 1 ; +∞ [, but the limits of u(x) at ±∞ are 0^+ (because ln1 = 0) ; hence u(x) is always Positive on]-∞ ; 0 [∪] 1 ; +∞ [, this proves that the function F is increasing on both intervals]-∞ ; 0 [and] 1 ; +∞ [.

x	- ∞		0 1		+ ∞
Sign [u'(x)]		⊕	||//////////||	—	
Variations & Sign of u(x)	0^+	↗ ⊕	||//////////||	↘ ⊕	0^+
Var. of F(x)	e^{-2}	↗	1^- ||//////////|| 0^+	↗	e^{-2}

5. **Study of the limits at the ends of the intervals]-∞ ; 0 [∪] 1 ; +∞ [**

   $(a)\ \lim_{x\to 0^-} F(x) = 1$ because $\lim_{x\to 0^-} x\ln\left(1-\frac{1}{x}\right) = \lim_{x\to 0^-}[x\ln(1-x) + (-x)\ln(-x)]\quad (x<0)$

   $= \lim_{x\to 0^-}[x\ln(1-x)] + \lim_{x\to 0^-}[(-x)\ln(-x)] = 0\times\ln 1 + 0 = 0$ (*See later that* $\lim_{X\to 0+}[X\ln X] = 0^-$)

   Then by continuity of the Exp. function $\lim_{X\to 0}[ExpX] = e^0 = 1 \therefore \lim_{x\to 0^-} F(x) = 1$

   $(b)\ \lim_{x\to 1^+} F(x) = 0$ because $\lim_{x\to 1^+} x\ln\left(1-\frac{1}{x}\right) = -\infty\ (x>1)$ *and* $\lim_{X\to -\infty}[ExpX] = 0^+ \therefore \lim_{x\to 1^+} F(x) = 0^+$

Hence to **extend the function F by continuity** at x = 0 and x = 1 we may fix F(0) = 1 and F(1) = 0.

$$(c)\ \lim_{x\to\pm\infty} F(x) = \frac{1}{e^2} \text{ because } \lim_{x\to\pm\infty} x\ln\left(1-\frac{1}{x}\right) = \lim_{x\to\pm\infty} -\frac{\ln\left(1+\frac{1}{(-x)}\right)}{\frac{1}{(-x)}} = -\lim_{X\to 0^\pm}\frac{\ln(1+X)}{X} = -\ln'(1) = -\frac{1}{1} = -1$$

$$\therefore \lim_{x\to\pm\infty} F(x) = \lim_{x\to\pm\infty} Exp2x\ln\left(1-\frac{1}{x}\right) = Exp(-2) = e^{-2} = \frac{1}{e^2} \simeq 0.14$$

(∴ *asymptote* : y= $\frac{1}{e^2}$ in +∞ and in-∞)

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Problem II.(5.2) : Let $F(x)=\left(\frac{1}{x}-1\right)^{2x}$

6. By definition the function F is composed of the two functions $f(x)=\left(\frac{1}{x}-1\right)$ and $u(x)=2x$

such that $F(x)=Exp\left[\ln[(f(x))^{u(x)}\right]=Exp\left(u(x)\ln[f(x)]\right)=e^{u(x)\ln[f(x)]}=e^{2x\ln\left(\frac{1}{x}-1\right)}$

Therefore *f(x)* must be strictly positive, which means that *0 < x < 1*

7. The derivative of F(x) is $F'(x)=[f(x)]^{u(x)}\left[u'(x).\ln[f(x)]+u(x).\frac{f'(x)}{f(x)}\right]$ $\because$ $ln'[f(x)]=\frac{f'(x)}{f(x)}$

$$F'(x)=\left(\frac{1}{x}-1\right)^{2x}\left[2.\ln\left(\frac{1}{x}-1\right)+2x.\frac{-\frac{1}{x^2}}{\frac{1-x}{x}}\right]=2\left(\frac{1}{x}-1\right)^{2x}\left[\ln\left(\frac{1}{x}-1\right)-\frac{1}{1-x}\right]$$

8. $Sgn[F'(x)]=Sgn\left[\ln\left(\frac{1}{x}-1\right)-\frac{1}{1-x}\right]$ for $0<x<1$. *Let* $u(x)=\ln\left(\frac{1}{x}-1\right)-\frac{1}{1-x}$

$$u'(x)=\frac{-\frac{1}{x^2}}{\frac{1-x}{x}}-\frac{1}{(x-1)^2}=-\frac{1}{x(x-1)^2}$$

Hence **u'(x) < 0 for 0 < x < 1**

9. **Study of the Variations** of F : The function u is decreasing on] 0 ; 1 [, but the limits of u(x) in 0^+ is +∞; and is - ∞ in 1^-. Hence the function u changes sign in one point a, on] 0 ; 1 [. This proves that the function F is increasing on]0 ; a] and decreasing on] a ; 1 [, with a maximum m = f(a).

x	- ∞	0		a		1	+ ∞
Sign [u'(x)]	//////////	‖		—		‖	//////////////////
Variations & Sign of u(x)	//////////	‖ + ∞	⊕	↘ 0	—	- ∞ ‖	//////////////////
Var. of F(x)	//////////	‖ 1^+	↗	m	↘	0^+ ‖	//////////////////

10. **Study of the limits at the ends of the interval] 0 ; 1 [**

$(a)\ \lim_{x\to0^+}F(x)=1$ because $\lim_{x\to0^+}x\ln\left(\frac{1}{x}-1\right)=\lim_{x\to0^+}[x\ln(1-x)-x\ln(x)]\quad(0<x<1)$

$=\lim_{x\to0^+}[x\ln(1-x)]-\lim_{x\to0^+}[x\ln x]=0\times\ln1-0^-=0^+\quad(\lim_{X\to0+}[X\ln X]=0^-)$

Then by continuity of the Exp. function $\lim_{X\to0}[ExpX]=e^0=1 \therefore \lim_{x\to0+}F(x)=1$

$(b)\ \lim_{x\to1^-}F(x)=0$ because $\lim_{x\to1^-}x\ln\left(\frac{1}{x}-1\right)=-\infty\ (0<x<1)$ *and* $\lim_{X\to-\infty}[ExpX]=0^+ \therefore \lim_{x\to1^-}F(x)=0^+$

11. **Graph of the function** $F(x)=\left|\frac{1}{x}-1\right|^{2x}$

It's the réunion of the graphs or the two previous function functions.
The approximate values of a and m = F(a), can be determined with a calulator.
The position of the tangent lines at (0;1) and at (1;0) is a more complicated question that will be studied later …