Euler 1707 - 1783

Euler's method of construction of the Exponential function

Objective : *build a function defined on* $\mathbb{R}$ *such that for any* $x \in \mathbb{R}$,

$$f'(x) = f(x) \text{ and } f(0) = 1$$

- Reminder : from the definition of the derivative of a function f in one point a :

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

$$\therefore \frac{f(a+h) - f(a)}{h} = f'(a) + \varepsilon(h) \quad \text{with} \quad \lim_{h \to 0} \varepsilon(h) = 0$$

$$f(a + h) = f(a) + h.f'(a) + h \cdot \varepsilon(h)$$

- *For any* small *value of h, close to 0,* $h \cdot \varepsilon(h)$ *is negligeable*

then $f(a + h) \approx f(a) + h.f'(a) + \ldots\ldots$

In this case we have $f'(a) = f(a)$ *and* $f(0) = 1$

then for a = 0

➔ $f(0 + h) \approx f(0) + h.f'(0) = f(0) + h.f(0) = 1 + h$

$f(h) \approx 1 + h$

➔ $f(2h) = f(h + h) \approx f(h) + h.f'(h)$

$f(2h) \approx f(h).(1+h) = (1+h)^2$

Let $u_n = f(nh)$

$f[(n+1)h] = f(nh + h) = f(nh)(1 + h)$

$u_{n+1} = u_n(1+h)$

The sequence (u_n) *is Geometric*

- 1^{st} *term* $u_0 = f(0) = 1$
- *reason* $q = 1+h$

$$u_n = f(nh) \approx (1 + h)^n$$

$$f(nh) \approx (1 + h)^n$$

for example let $h = \frac{1}{100}$ *and n = 100 then*

$$f(1) = f\left(100 \times \frac{1}{100}\right) \approx \left(1 + \frac{1}{100}\right)^{100} \approx 2.7..$$

$$f(2) = f\left(200 \times \frac{1}{100}\right) \approx \left(1 + \frac{1}{100}\right)^{200} = \left[\left(1 + \frac{1}{100}\right)^{100}\right]^2 \approx (2.7)^2 \approx 7.3$$

then for any Integer k

$$f(k) = f\left(k \times 100 \times \frac{1}{100}\right) \approx \left(1 + \frac{1}{100}\right)^{k \times 100} = \left[\left(1 + \frac{1}{100}\right)^{100}\right]^k \approx (2.7)^k$$

EULER was able to prove that the sequence defined by $v_n = \left(1 + \frac{1}{n}\right)^n$ has a finite limit which he named e, with $e \approx 2.718\ldots$

then for n = 100 we have : $\left(1 + \frac{1}{100}\right)^{100} \approx e \approx 2.718$

and for any Integer k we have : $f(k) = e^k$

$$\because f(k) = f\left(k \times 100 \times \frac{1}{100}\right) \approx \left(1 + \frac{1}{100}\right)^{k \times 100} = \left[\left(1 + \frac{1}{100}\right)^{100}\right]^k \approx e^k$$

Question : *if x is any Real number can we write* $f(x) = e^x$ *???*

If n is a very large number compared to x, then we could

let $h = \frac{x}{n}$ and $\frac{n}{x}$ be a large number

$$f(x) = f(n \times \frac{x}{n}) \approx \left(1 + \frac{x}{n}\right)^n = \left[\left(1 + \frac{1}{\frac{n}{x}}\right)^{\frac{n}{x}}\right]^x \approx e^x$$

Problem : the formula : $\left(1 + \frac{x}{n}\right)^n = \left[\left(1 + \frac{1}{\frac{n}{x}}\right)^{\frac{n}{x}}\right]^x$

is "*wrong*" …

The formula : $\left(1 + \frac{x}{n}\right)^n = \left[\left(1 + \frac{1}{\frac{n}{x}}\right)^{\frac{n}{x}}\right]^x$

is "*wrong*" …

because it was established that

$$\left(a^m\right)^n = a^{m \times n}$$

only for the Integers m and n …

…not for Real numbers … yet !)

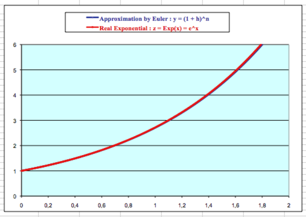
So what ? ! ?

Let's look at the present situation
more closely with the computer…

(Euler had a fabulous capacity of calculations
and did not need to use a computer …)

h = 0,01

n	x	y = (1 + h)^n	z = Exp(x)
0	0	1,0000000	1
1	0,01	1,0100000	1,010050167
2	0,02	1,0201000	1,02020134
3	0,03	1,0303010	1,030454534
4	0,04	1,0406040	1,040810774
5	0,05	1,0510101	1,051271096
6	0,06	1,0615202	1,061836547
7	0,07	1,0721354	1,072508181
8	0,08	1,0828567	1,083287068
9	0,09	1,0936853	1,094174284
10	0,1	1,1046221	1,105170918
11	0,11	1,1156683	1,11627807
12	0,12	1,1268250	1,127496852
13	0,13	1,1380933	1,138828383
14	0,14	1,1494742	1,150273799
15	0,15	1,1609690	1,161834243
16	0,16	1,1725786	1,173510871
17	0,17	1,1843044	1,185304851
18	0,18	1,1961475	1,197217363
19	0,19	1,2081090	1,209249598
20	0,2	1,2201900	1,221402758
21	0,21	1,2323919	1,23367806
22	0,22	1,2447159	1,246076731
23	0,23	1,2571630	1,25860001
24	0,24	1,2697346	1,27124915
25	0,25	1,2824320	1,284025417
26	0,26	1,2952563	1,296930087
27	0,27	1,3082089	1,309964451
28	0,28	1,3212910	1,323129812
29	0,29	1,3345039	1,336427488
30	0,3	1,3478489	1,349858808
31	0,31	1,3613274	1,363425114
32	0,32	1,3749407	1,377127764
33	0,33	1,3886901	1,390968128
34	0,34	1,4025770	1,404947591
35	0,35	1,4166028	1,419067549
36	0,36	1,4307688	1,433329415
37	0,37	1,4450765	1,447734615
38	0,38	1,4595272	1,462284589
39	0,39	1,4741225	1,476980794
40	0,4	1,4888637	1,491824698
41	0,41	1,5037524	1,506817785
42	0,42	1,5187899	1,521961556
43	0,43	1,5339778	1,537257524
44	0,44	1,5493176	1,552707219
45	0,45	1,5648107	1,568312185
46	0,46	1,5804589	1,584073985
47	0,47	1,5962634	1,599994193
48	0,48	1,6122261	1,616074402
49	0,49	1,6283483	1,63231622
50	0,5	1,6446318	1,648721271
51	0,51	1,6610781	1,665291195
52	0,52	1,6776889	1,68202765
53	0,53	1,6944658	1,698932309
54	0,54	1,7114105	1,716006862
55	0,55	1,7285246	1,733253018
56	0,56	1,7458098	1,7506725
57	0,57	1,7632679	1,768267051
58	0,58	1,7809006	1,786038431
59	0,59	1,7987096	1,803988415
60	0,6	1,8166967	1,8221188
61	0,61	1,8348637	1,840431399
62	0,62	1,8532123	1,858928042



In observing the spread sheet, we can see that for any real number x written in decimal form we can find a number N such that :

$$x \approx N \times \frac{1}{100} \quad \text{or} \quad x \approx N \times \frac{1}{1000} \quad \text{or} \quad x \approx N \times \frac{1}{10^p}$$

For instance : $x = 0.67 = 67 \times \frac{1}{100}$

then $f(x) \approx \left(1 + \frac{1}{100}\right)^{67} \approx 1{,}95$

We may decide to write Exp(0.67) or $e^{0.67}$ *to match the previous results for the integers.*

So what ? ! ? …

Euler *was able to calculate these values with a great acuracy, but he was mainly able to prove that the function defined by the original conditions* $f'(x) = f(x)$ *and* $f(0) = 1$ *is defined for any real number and is continuous.*

(since it has a derivative).

He named it "Exponential" and he was also able to prove that this new function Exp. had a the fundamental property of transforming sums into products for any real numbers u and v

Exp (u + v) = Exp(u) • Exp(v)

Or $e^{u+v} = e^u . e^v$

Then the previous formula makes sense.

Demo of the fundamental formula of the Exponential function (1)

Let's prove that if

$f'(x) = f(x)$ *and* $f(0) = 1$

then for any real numbers u and v we can write

$f(u + v) = f(u).f(v)$

Demo of the fundamental formula of the Exponential function (2)

Let $F(x) = f(a+x).f(-x)$

Then $F'(x) = f'(a+x).f(-x) + f(a+x).[-f'(-x)]$

$= f(a+x)\bullet[f(-x)-f(-x)] = 0$

Then F(x) is a constant :

$F(x)=F(0)=f(a) \Rightarrow f(x+a).f(-x) = f(a)$

And particularly for $a = 0$

$f(x).f(-x) = f(0) = 1$

Demo of the fundamental formula of the Exponential function (3)

Then by multiplying both members by f(x) in the previous equation we have :

$f(a+x).f(-x).f(x)=f(a).f(x)$

hence : $f(a+x) = f(a).f(x)$

Or $f(u + v) = f(u).f(v)$

Demo of the fundamental formula of the Exponential function (4)

Then if we use the exponential notation :

$Exp(u + v) = Exp(u).Exp(v)$

$$e^{u+v} = e^u.e^v$$

More over Euler established a fundamental formula of development of functions in "power series" such that :

$$e^x = \frac{x^0}{0!} + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!} + \ldots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

It is this formula (using at least 6 or 7 terms) that is used in computers and pocket calculators to provide us with the "exact" values of Exp(x).
Then we can check that the values found by the differential relationship are correct approximate values.

谢谢