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Name : [ ] Grade : [ ]

Senior 1.4 – TEST [B] - May 27^{th} – 60 min.

Recurrence in Sequences & Series

I. The turtle and the dog

D <…………..100 m ……………….> T

The dog (D) runs 10 times faster than the turtle (T). They start at a distance of 100 m. but every time the dog runs the distance between him and the turtle, the turtle continues to walk … therefore the distance left between them is 10 m, and so on….

Let d_0 be the initial distance between D and T,

Let d_1 be the distance left between D and T after the first run,

Let d_n the distance left between D and T the after the n^{th} run

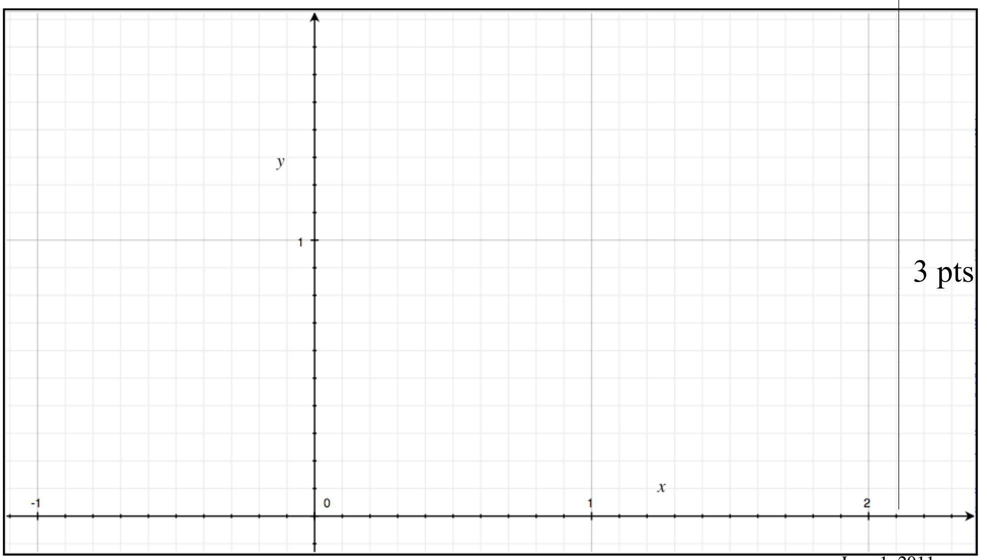
1. Write the relation ship between d_n and d_{n+1}: 1 pt
2. Find the total distance run by the dog after the n^{th} run : $S_n = \sum_{k=0}^{k=n-1} d_k =$ 2 pts
3. Find the total distance run by the dog to get the turtle $S = \sum_{n=0}^{\infty} d_n =$ 2 pts

II. Fibonacci for ever … $\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\ldots\ldots+\sqrt{1}}}}}}}}}}$

Let f be the function defined by $f(x) = \sqrt{1+x}$, (x >-1) and (u_n) the sequence defined by

$$u_{n+1} = \sqrt{1+u_n} \quad \text{and} \quad u_0 = 0$$

1. Write the expression of u_1, u_2, u_3 1 pts
2. Show that the function f is increasing on [0 ; +∞[1 pts
3. Prove by recurrence that for any $n \geq 0$, $0 \leq u_n \leq 2$ 3 pts
4. Prove by recurrence that for any $n \geq 0$, $u_n \leq u_{n+1}$ 3 pts
5. Explain why the sequence (u_n) has a finite limit x. 2 pts
6. Show how to find the value of x 2 pts
7. Graph the function f and the construction of the sequence 3 pts



III. Adjacent Sequences and approximation of the square root of a number

Let f be defined by $f(x)=\dfrac{x+5}{x+1}$ and the sequence (x_n) defined by $x_{n+1}=f(x_n)$ *with* $x_0=1$

We are going to study two different methods to find the limit of this sequence :

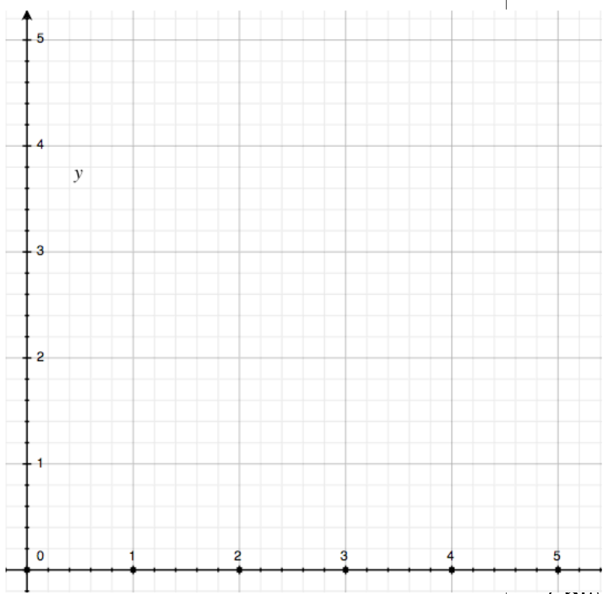
1. First method :
   a. Prove by recurrence *that for any* $n \geq 0$, $1 \leq x_n \leq 3$ — 2 pts

   b. Show that for any $n \geq 0$,
   $$\left|x_n-\sqrt{5}\right| \leq q\left|x_{n-1}-\sqrt{5}\right| \quad \text{with } q=\frac{2}{1+\sqrt{5}}$$

   c. Prove by recurrence that for any $n \geq 0$,
   $$0 \leq \left|x_n-\sqrt{5}\right| \leq 2.q^n$$

   d. Explain what is the limit of (x_n)

   e. Show the construction on the graph ==> — 2 pts



2. Second method : *using the variations of the function f and* $g=f\circ f$, with $u_n=x_{2n}=g(u_{n-1})$ and $v_n=x_{2n+1}=g(v_{n-1})$
   a. Explain why g is an increasing function on [0 ; +∞[and calculate $g(x)=f[f(x)]$. — 2 pts

   b. Prove by recurrence that the sequence *(u_n) is increasing (*we admit that *(v_n)* is *decreasing)* — 2 pts

   c. Show that for any $n \geq 0$, $\left|v_{n+1}-u_{n+1}\right| \leq \frac{1}{4}\left|v_n-u_n\right|$ — 2 pts

   d. Prove by recurrence that for any $n \geq 0$, $\left|v_n-u_n\right| \leq 2\left(\frac{1}{4}\right)^n$ — 2 pts

   e. Explain why the two sequences *(u_n)* and *(v_n)* are *adjacent* — 1 pt

   f. Show what is the common limit of *(u_n), (v_n)* and *(x_n)*. — 1 pt

   g. Use u_3 and v_3 to give an approximate value of √5. — 1pt