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ANSWERS

Senior 1.4 – TEST [A] - May 27th – 60 min.

Recurrence in Sequences & Series

I. The turtle and the dog

D <…………..100 m ……………….> T

The dog (D) runs 10 times faster than the turtle (T). They start at a distance of 100 m. but every time the dog runs the distance between him and the turtle, the turtle continues to walk … therefore the distance left between them is 10 m, and so on….

Let d_0 be the initial distance between D and T,

Let d_1 be the distance left between D and T after the first run,

Let d_n the distance left between D and T the after the n^{th} run

1. Relationship between d_n and d_{n+1}: $d_{n+1} = \frac{1}{10} d_n$ geom. sequ., reason : $q = \frac{1}{10}$ — 1 pt
2. Find the total distance run by the dog after the n^{th} run : — 2 pts
3. Find the total distance run by the dog to get the turtle .... — 2 pts

$$S_n = \sum_{k=0}^{k=n-1} d_k = d_0 \frac{1-q^n}{1-q} = 100 \frac{1-\left(\frac{1}{10}\right)^n}{1-\frac{1}{10}} = \frac{1000}{9}\left(1-10^{-n}\right) \Rightarrow S = \sum_{n=0}^{\infty} d_n = \frac{d_0}{1-q} = \frac{100}{1-\frac{1}{10}} = \frac{1000}{9} = 111{,}111...$$

II. Fibonacci for ever … $\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+.......+\sqrt{1}}}}}}}}}}$

Let f be the function defined by $f(x) = \sqrt{1+x}$, (x >-1) and $u_{n+1} = \sqrt{1+u_n}$ and $u_0 = 0$

1. Expression of u_1, u_2, u_3 : $u_1 = \sqrt{1},\ u_2 = \sqrt{1+\sqrt{1}} = 2,\ u_3 = \sqrt{1+\sqrt{1+\sqrt{1}}} = \sqrt{1+\sqrt{2}}$ — 1 pts
2. f is increasing on [0 ; +∞[: $0 \le x_1 \le x_2 \Rightarrow 1 \le x_1+1 \le x_2+1 \Rightarrow 1 \le \sqrt{x_1+1} \le \sqrt{x_2+1} \Rightarrow f(x_1) \le f(x_2)$ — 1 pts
3. Prove by recurrence that for any $n \ge 0$, $0 \le u_n \le 2$ [P_n] — 3 pts
   - i. Initialization : [P_0] ⇔ $0 \le u_0 \le 2$ ⇔ $0 \le 0 \le 2$ TRUE !
   - ii. Heredity : [P_n] ⇔ $0 \le u_n \le 2$ ⇒ *[f increasing]* $f(0) \le f(u_n) \le f(2)$ ⇔ $1 \le u_{n+1} \le \sqrt{1+\sqrt{2}} \le 2$ ⇒ [P_{n+1}]
4. Prove by recurrence that for any $n \ge 0$, $u_n \le u_{n+1}$ [P_n] — 3 pts
   - i. Initialization : [P_0] ⇔ $u_0 \le u_1$ ⇔ $0 \le 1$ TRUE !
   - ii. Heredity : [P_n] ⇔ $u_n \le u_{n+1}$ ⇒ *[f increasing]* $f(u_n) \le f(u_{n+1})$ ⇒ $u_{n+1} \le u_{n+2}$ ⇒ [P_{n+1}]
5. Explain why the sequence (u_n) has a finite limit x. — 2 pts

   Because (u_n) is **increasing** and has a **Majorant** M=2 ∴ (u_n) has a limit $x \le$ M
6. How to find the value of x : — 2 pts

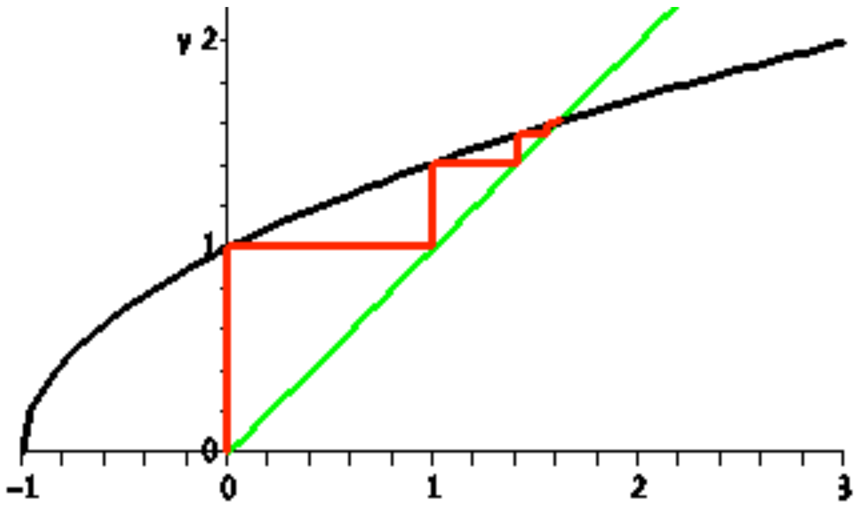
   lim $u_n = x$ and lim $u_{n+1} = x$

   and lim $f(u_n)$ = f(lim u_n)

   ∴ $x = f(x) = \sqrt{x+1}$

   ⇔ $x^2 - x - 1 = 0$ (x ≥ 0)

   ⇔ $x = \frac{1+\sqrt{5}}{2}$ *(Fibonacci)*
7. Graph the function f and the construction of the sequence — 3 pts



III. Adjacent Sequences and approximation of the square root of a number

Let f be defined by $f(x)=\dfrac{x+3}{x+1}$ and the sequence (x_n) defined by $x_{n+1}=f(x_n)$ *with* $x_0=0$

We are going to study two different methods to find the limit of this sequence :

1. First method :

a. Prove by recurrence *that for any* $n\geq 0$, $0\leq x_n\leq 3$ [P_n] $f(x)=\dfrac{x+3}{x+1}=1+\dfrac{2}{x+1}$ $\therefore$ *f Decreasing* — 2 pts

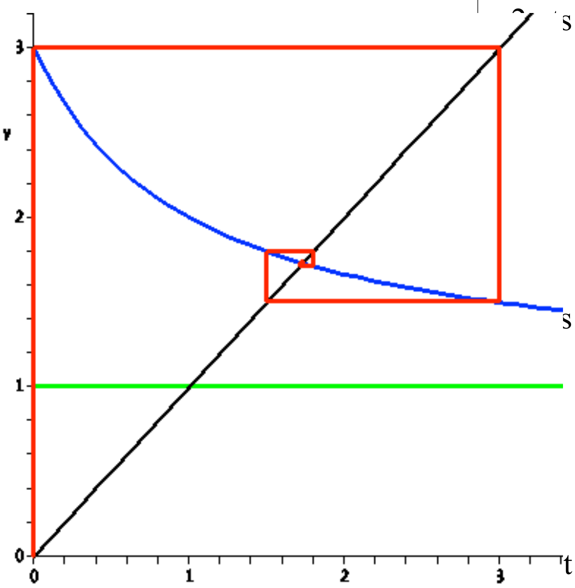
i. Initialization : [P_0] $\Leftrightarrow$ $0\leq x_0\leq 3$ $\Leftrightarrow$ $0\leq 0\leq 3$ TRUE !

ii. Heredity : [P_n] $\Leftrightarrow$ $0\leq x_n\leq 3 \Rightarrow$ *[f DEcreasing]* $f(0)\geq f(x_n)\geq f(3)$ $\Rightarrow 3\leq x_{n+1}\geq 1.5 \Rightarrow$ [P_{n+1}]

i. For any $n\geq 0$, $\left|x_n-\sqrt{3}\right|\leq q\left|x_{n-1}-\sqrt{3}\right|$ *with* $q=\dfrac{2}{1+\sqrt{3}}=\sqrt{3}-1$

$$\left|x_n-\sqrt{3}\right|=\left|\frac{x_{n-1}+3}{x_{n-1}+1}-\sqrt{3}\right|=\left|\frac{(x_{n-1}-\sqrt{3})(1-\sqrt{3})}{x_{n-1}+1}\right|$$

$$\therefore\ \left|x_n-\sqrt{3}\right|\leq\frac{\sqrt{3}-1}{x_{n-1}+1}\left|x_{n-1}-\sqrt{3}\right|\leq(\sqrt{3}-1)\left|x_{n-1}-\sqrt{3}\right|\ \because\ x_{n-1}\geq 0$$



b. Prove by recurrence that for any $n\geq 0$,
$0\leq\left|x_n-\sqrt{3}\right|\leq q^n\sqrt{3}$ [P_n]

i. Initialization : [P_0] $\Leftrightarrow$ $0\leq\left|0-\sqrt{3}\right|\leq q^0\sqrt{3}$ TRUE !

ii. Heredity : [P_n] $\Leftrightarrow$ $0\leq\left|x_n-\sqrt{3}\right|\leq q^n\sqrt{3}$

$$\left|x_{n+1}-\sqrt{3}\right|\leq q\left|x_n-\sqrt{3}\right|\Rightarrow\left|x_{n+1}-\sqrt{3}\right|\leq q^{n+1}\sqrt{3}\Leftrightarrow[P_{n+1}]$$

c. Explain what is the limit of (x_n)
$|q|<1$ $\Rightarrow$ $\lim|x_n-\sqrt{3}|=0$ $\Leftrightarrow$ $\lim x_n=\sqrt{3}$

2. Second method : *using the variations of the function f and* $g=f\circ f$, with $u_n=x_{2n}=g(u_{n-1})$ and $v_n=x_{2n+1}=g(v_{n-1})$

a. *f* is **DE**creasing on [0 ; +∞[and $\therefore$ $g=f\circ f$ *is* **IN***creasing and* $g(x)=f(f(x))=1+\dfrac{2}{f(x)+1}=\dfrac{2x+3}{x+2}=2-\dfrac{1}{x+2}$ — 2 pts

b. Prove by recurrence that the sequence *(u_n) is increasing (*we admit that *(v_n)* is *decreasing)* — 2 pts

i. Initialization : [P_0] $\Leftrightarrow$ $u_0\leq u_1$ $\Leftrightarrow$ $0\leq 3$ TRUE !

ii. Heredity : [P_n] $\Leftrightarrow$ $u_n\leq u_{n+1}\Rightarrow$ *[g INcreasing]* $g(u_n)\leq g(u_{n+1})\Rightarrow u_{n+1}\leq u_{n+2}\Rightarrow$ [P_{n+1}]

c. For $n\geq 0$, $\left|v_{n+1}-u_{n+1}\right|=\left|g(v_n)-g(u_n)\right|=\left|\dfrac{-1}{v_n+2}+\dfrac{1}{u_n+2}\right|=\dfrac{\left|v_n-u_n\right|}{(v_n+2)(u_n+2)}\leq\dfrac{1}{4}\left|v_n-u_n\right|$ $\because$ $u_n\geq 0\ \&\ v_n\geq 0$ — 2 pts

d. Prove by recurrence that for any $n\geq 0$, $\left|v_n-u_n\right|\leq 3\left(\dfrac{1}{4}\right)^n$ [P_n] — 2 pts

i. Initialization : [P_0] $\Leftrightarrow$

ii. $[P_{n+1}]$ $\left|v_n-u_n\right|\leq 3\left(\dfrac{1}{4}\right)^n$ & $\left|v_{n+1}-u_{n+1}\right|\leq\dfrac{1}{4}\left|v_n-u_n\right|\Rightarrow\left|v_{n+1}-u_{n+1}\right|\leq\dfrac{1}{4}3\left(\dfrac{1}{4}\right)^n\Rightarrow\left|v_{n+1}-u_{n+1}\right|\leq 3\left(\dfrac{1}{4}\right)^{n+1}$ $[P_{n+1}]$

e. Explain why the two sequences *(u_n)* and *(v_n)* are *adjacent* — 1 pt
(u_n) ↗ , (v_n) ↘ and $\lim|v_n-u_n|=0$ $\therefore$ ($|1/4|<1$) $\therefore$ $\lim(u_n)=\lim(v_n)=x$

f. Show what is the common limit of *(u_n), (v_n)* and *(x_n)*. — 1 pt
$\lim(u_n)=\lim(v_n)=x\Rightarrow\lim(x_n)=\lim(x_{n+1})=x$ *and* $\lim(x_{n+1})=f(\lim(x_n))$ (because f is continuous)

$$\therefore\ x=f(x)=\frac{x+3}{x+1}\Leftrightarrow x^2=3\ (x>0)\Leftrightarrow x=\sqrt{3}$$

g. Use u_3 and v_3 to give an approximate value of √3. — 1pt

$$u_3\leq\sqrt{3}\leq v_3\ \Rightarrow\ \boxed{\sqrt{3}\approx\frac{u_3+v_3}{2}=\frac{1{,}730+1{,}732}{2}=1.731...}$$ (error less than 10^{-3}).