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Name : ______ . Grade : ______

Senior 1.4 – TEST [A] - May 27th – 60 min.

Recurrence in Sequences & Series

I. The turtle and the dog

D <…………..100 m ……………..> T

The dog (D) runs 10 times faster than the turtle (T). They start at a distance of 100 m. but every time the dog runs the distance between him and the turtle, the turtle continues to walk … therefore the distance left between them is 10 m, and so on….

Let d_0 be the initial distance between D and T,
Let d_1 be the distance left between D and T after the first run,
Let d_n the distance left between D and T the after the n^{th} run

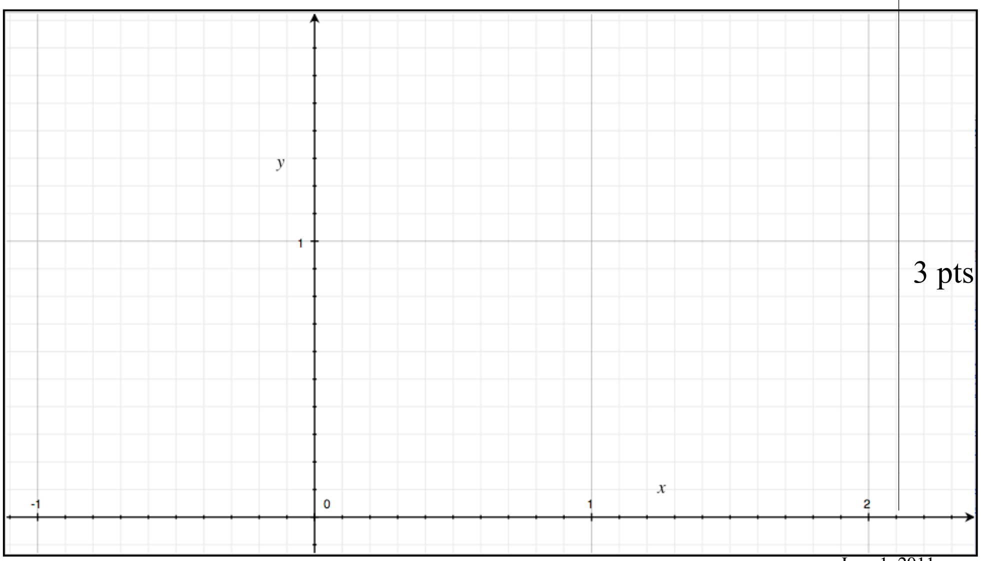
1. Write the relation ship between d_n and d_{n+1}: — 1 pt
2. Find the total distance run by the dog after the n^{th} run : $S_n = \sum_{k=0}^{k=n-1} d_k =$ — 2 pts
3. Find the total distance run by the dog to get the turtle $\boxed{S = \sum_{n=0}^{\infty} d_n = \quad}$ — 2 pts

II. Fibonacci for ever … $\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+.......+\sqrt{1}}}}}}}}}}$

Let f be the function defined by $f(x) = \sqrt{1+x}$, (x >-1) and (u_n) the sequence defined by

$$u_{n+1} = \sqrt{1+u_n} \quad \text{and} \quad u_0 = 0$$

1. Write the expression of u_1, u_2, u_3 — 1 pts
2. Show that the function f is increasing on [0 ; +∞[— 1 pts
3. Prove by recurrence that for any $n \geq 0$, $0 \leq u_n \leq 2$ — 3 pts
4. Prove by recurrence that for any $n \geq 0$, $u_n \leq u_{n+1}$ — 3 pts
5. Explain why the sequence (u_n) has a finite limit x. — 2 pts
6. Show how to find the value of x — 2 pts
7. Graph the function f and the construction of the sequence — 3 pts



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III. Adjacent Sequences and approximation of the square root of a number

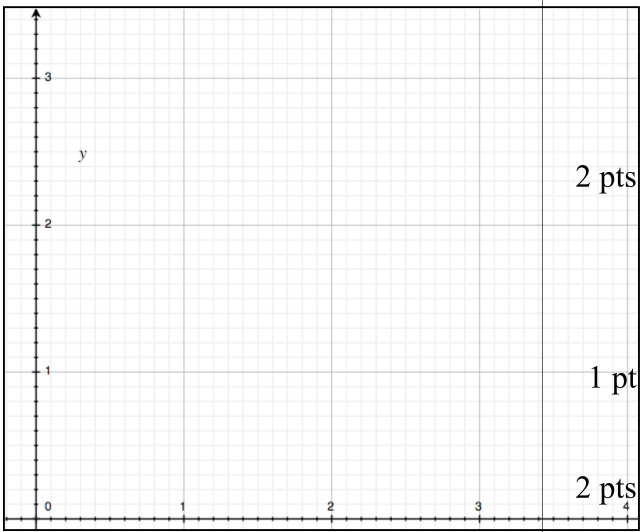
Let f be defined by $f(x)=\dfrac{x+3}{x+1}$ and the sequence (x_n) defined by $x_{n+1}=f(x_n)$ *with* $x_0=0$

We are going to study two different methods to find the limit of this sequence :

1. First method :

a. Prove by recurrence *that for any* $n \geq 0$, $0 \leq x_n \leq 3$ — 2 pts

b. Show that for any $n \geq 0$, $\left|x_n-\sqrt{3}\right| \leq q\left|x_{n-1}-\sqrt{3}\right|$ *with* $q=\dfrac{2}{1+\sqrt{3}}$ — 2 pts



c. Prove by recurrence that for any $n \geq 0$,
$0 \leq \left|x_n-\sqrt{3}\right| \leq q^n\sqrt{3}$ — 2 pts

d. Explain what is the limit of (x_n) — 1 pt

e. Show the construction on the graph ==> — 2 pts

2. Second method : *using the variations of the function f and* $g=f\circ f$, with $u_n=x_{2n}=g(u_{n-1})$ and $v_n=x_{2n+1}=g(v_{n-1})$

a. Explain why g is an increasing function on $[0\ ;+\infty[$ and calculate $g(x)=f[f(x)]$. — 2 pts

b. Prove by recurrence that the sequence (u_n) *is increasing* (we admit that (v_n) is *decreasing)* — 2 pts

c. Show that for any $n \geq 0$, $\left|v_{n+1}-u_{n+1}\right| \leq \dfrac{1}{4}\left|v_n-u_n\right|$ — 2 pts

d. Prove by recurrence that for any $n \geq 0$, $\left|v_n-u_n\right| \leq 3\left(\dfrac{1}{4}\right)^n$ — 2 pts

e. Explain why the two sequences (u_n) and (v_n) are *adjacent* — 1 pt

f. Show what is the common limit of (u_n), (v_n) and (x_n). — 1 pt

g. Use u_3 and v_3 to give an approximate value of $\sqrt{3}$. — 1pt