General Theorems of Convergence of Monotonous Sequences

Theorem I : If (u_n) is monotonous and bounded then (u_n) has a finite limit.

Proof : Let (u_n) be an increasing sequence such that for any $n \geq n_0$, $u_n \leq M$

[M is called a Majorant of (u_n)]

$$u_0 \leq u_1 \leq u_2 \leq u_3 \leq \ldots\ u_n \leq u_{n+1} \leq \ldots\ldots \leq M$$

i. If (u_n) has at least one Majorant M, then any number M' $\geq$ M is also a Majorant of (u_n). Therefore the set E of majorants of (u_n) is a set of Real numbers which are all $\geq u_{n0}$. Hence that set E has one **smallest element** α (we admit this result) :

$$u_0 \leq u_1 \leq u_2 \leq u_3 \leq \ldots\ u_n \leq u_{n+1} \leq \ldots \leq \alpha \leq \ldots \leq M \leq \ldots \leq M'$$

ii. Let's prove that this smallest element α is the limit of (u_n) :

Then for any $\varepsilon > 0$ (as small as we want), we must have one rank $N > 0$, such that [*otherwise α would not be the smallest <u>Majorant</u> of (u_n)*]

$$u_0 \leq u_1 \leq u_2 \leq u_3 \leq \ldots\ldots(\alpha\text{-}\varepsilon\,) \leq\ u_N \leq u_{N+1} \leq \ldots \leq \alpha \leq \ldots \leq M$$

This is by definition showing that **lim $(u_n) = \alpha$**

iii. NB : We have proved that the sequence (u_n) has a finite limit, but we don't know the value of that limit ! We can only say that **lim $(u_n) = \alpha \leq M$**

iv. A similar proof applies to sequences that are decreasing and have a Minorant m.

Theorem II : If (v_n) and (w_n) are two ADJACENT sequences, they both CONVERGE to a same limit.

Definition : two sequences (v_n) and (w_n) are said to be **ADJACENT** if and only if, they have the two following properties :

i. (w_n) is increasing and (v_n) is decreasing
ii. $\lim |v_n - w_n| = 0$

$$w_0 \leq w_1 \leq w_2 \leq w_3 \leq \ldots\ w_n \leq w_{n+1} \leq \ldots\ldots \leq v_{n+1} \leq v_n \leq \ldots v_3 \leq v_2 \leq \ldots\ v_1 \leq v_0$$

Proof : from the previous theorem we can tell that both (v_n) and (w_n) have a limit, because they are both monotonous and bounded : (w_n) has at least one <u>*majorant*</u> : v_0, and (v_n) has at least one <u>*minorant*</u> : w_0.

Let $\alpha = \lim\ (v_n)$ and β lim (v_n). , then because of condition (ii) we have $\alpha = \beta$.

Since if $\alpha \neq \beta$ then the difference of $|v_n - w_n|$ could not be zeroed.

$$w_0 \leq w_1 \leq w_2 \leq w_3 \leq \ldots\ w_n \leq w_{n+1} \leq \ldots\alpha \leq \ldots \leq \beta\ \ldots \leq v_{n+1} \leq v_n \leq \ldots v_3 \leq v_2 \leq \ldots\ v_1 \leq v_0$$

Finally we have $\lim\ (v_n) = \alpha = \beta = \lim\ (w_n)$.

NB : this is not providing the value of that common limit. we can only say that it's between v_0 and w_0.