http://beijingshanmaths.org
jiguanglaoshi@gmail.com



# General Theorems of Convergence of Monotonous Sequences

**Theorem I** : If $(u_n)$ is monotonous and bounded then $(u_n)$ has a finite limit.

*Proof* : Let $(u_n)$ be an increasing sequence such that for any $n \geq n_0$ , $u_n \leq M$
*[M is called a Majorant of $(u_n)$]*

$$u_0 \leq u_1 \leq u_2 \leq u_3 \leq \dots u_n \leq u_{n+1} \leq \dots\dots \leq M$$

i. If $(u_n)$ has at least one Majorant M, then any number $M' \geq M$ is also a Majorant of $(u_n)$. Therefore the set E of majorants of $(u_n)$ is a set of Real numbers which are all $\geq u_{n0.}$ Hence that set E has one **smallest element** $\alpha$ (we admit this result) :

$$u_0 \leq u_1 \leq u_2 \leq u_3 \leq \dots u_n \leq u_{n+1} \leq \dots \leq \alpha \leq \dots \leq M \leq \dots \leq M'$$

ii. Let's prove that this smallest element $\alpha$ is the limit of $(u_n)$ :
Then for any $\varepsilon > 0$ (as small as we want), we must have one rank $N > 0$,
such that [*otherwise $\alpha$ would not be the smallest Majorant of $(u_n)$* ]

$$u_0 \leq u_1 \leq u_2 \leq u_3 \leq \dots\dots(\alpha\text{-}\varepsilon) \leq u_N \leq u_{N+1} \leq \dots \leq \alpha \leq \dots \leq M$$

This is by definition showing that **lim $(u_n) = \alpha$**

iii. NB : We have proved that the sequence $(u_n)$ has a finite limit, but we don't know the value of that limit ! We can only say that **lim $(u_n) = \alpha \leq M$**

iv. A similar proof applies to sequences that are decreasing and have a Minorant m.

---

**Theorem II** : If $(v_n)$ and $(w_n)$ are two ADJACCENT sequences, they both CONVERGE to a same limit.

**Definition** : two sequences $(v_n)$ and $(w_n)$ are said to be **ADJACCENT** if and only if, they have the two following properties :

i. $(w_n)$ is increasing and $(v_n)$ is decreasing
ii. $\lim |v_n - w_n| = 0$

$$w_0 \leq w_1 \leq w_2 \leq w_3 \leq \dots w_n \leq w_{n+1} \leq \dots\dots \leq v_{n+1} \leq v_n \leq \dots v_3 \leq v_2 \leq \dots v_1 \leq v_0$$

**Proof :** from the previous theorem we can tell that both $(v_n)$ and $(w_n)$ have a limit, because they are both monotonous and bounded : $(w_n)$ has at least one *majorant* : $v_0$, and $(v_n)$ has at least one *minorant* : $w_0$.
Let $\alpha = \lim$ *$(v_n)$* and $\beta$ lim *$(v_n)$*. , then because of condition (ii) we have $\alpha = \beta$.
Since if $\alpha \neq \beta$ then the difference of $|v_n - w_n|$ could not be zeroed.

$$w_0 \leq w_1 \leq w_2 \leq w_3 \leq \dots w_n \leq w_{n+1} \leq \dots \alpha \leq \dots \leq \beta \dots \leq v_{n+1} \leq v_n \leq \dots v_3 \leq v_2 \leq \dots v_1 \leq v_0$$

Finally we have lim *$(v_n)$* $= \alpha = \beta =$ lim *$(w_n)$*.

*NB : this is not providing the value of that common limit. we can only say that it's between* $v_0$ and $w_{0.}$