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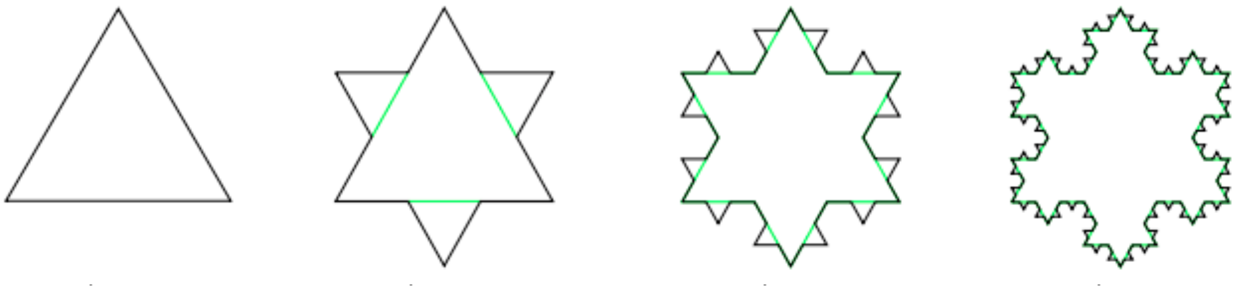
Geometric Series in Geometry (2)
Von Koch's snow flake

I. Construction of the basic pattern :

1. Divide the sides of an equilateral triangle into three equal parts
2. Erase the middle part of each side and replace it by an equilateral triangle
3. Do the same on each of the 4 new sides of the side …



n = 0 n=1 n=2 n=3

II. A paradoxal situation : we want to show that the length of the curve goes to infinity, while the area inside has a finite limit.

i. Let's call P_n the length of the perimeter of the n^{th} picture
ii. Let's call C_n the number of sides
iii. Let's call A_n the area enclosed in the n^{th} picture.

1. Write P_0, C_0, A_0 *(the length of the side of the initial triangle is 1 unit)*
$P_0 = 3$; $C_0 = 3$; $A_0 = \sqrt{3}/4$
2. Write P_1, C_1, A_1 , then P_2, C_2, A_2 :

$$P_1 = \frac{4}{3}P_0 = 4 \ ; \ C_1 = 4C_0 = 12 \ ; \ A_1 = A_0 + 3\left(\frac{1}{3}\right)^2 \frac{\sqrt{3}}{4} \ ; P_2 = \frac{4}{3}P_1 = \frac{16}{3} \ ; \ C_2 = 4C_1 = 48 \ ; \ A_2 = A_1 + C_1 \times \left[\left(\frac{1}{3}\right)^2\right]^2 \times \frac{\sqrt{3}}{4}$$

3. Write the relationship between P_{n+1} and P_n ; C_{n+1} and C_n ; A_{n+1} and A_n

$$P_{n+1} = \frac{4}{3}P_n \ ; \ C_{n+1} = 4C_n \ ; \ A_{n+1} = A_n + C_n\left[\left(\frac{1}{3}\right)^{n+1}\right]^2 \frac{\sqrt{3}}{4} \Rightarrow P_n = 3\left(\frac{4}{3}\right)^n \ ; \ C_n = 3\times 4^n \ ; A_{n+1} = A_n + 3\times 4^n\left(\frac{1}{3}\right)^{2(n+1)}\frac{\sqrt{3}}{4} = A_n + \frac{1}{3}\left(\frac{4}{9}\right)^n \frac{\sqrt{3}}{4}$$

4. Find the limits of each sequence (P_n) , (C_n) , (A_n) :

$\lim P_n = +\infty$ and $\lim C_n = +\infty$ (Geometric sequences, with reasons > 1)

The Sequence (A_n) is not geometric. It's the sum of A_0 and a geometric series :

$$A_n = A_{n-1} + \frac{1}{3}\left(\frac{4}{9}\right)^{n-1}\frac{\sqrt{3}}{4} = A_0 + \frac{1}{3}\frac{\sqrt{3}}{4} + \frac{1}{3}\left(\frac{4}{9}\right)^1\frac{\sqrt{3}}{4} + \ldots + \frac{1}{3}\left(\frac{4}{9}\right)^{n-1}\frac{\sqrt{3}}{4}$$

$$A_n = A_0 + \frac{1}{3}\frac{\sqrt{3}}{4}\left[1 + \left(\frac{4}{9}\right) + \left(\frac{4}{9}\right)^2 + \ldots + \left(\frac{4}{9}\right)^{n-1}\right] = A_0 + \frac{\sqrt{3}}{12}\sum_{k=0}^{k=n-1}\left(\frac{4}{9}\right)^k = A_0 + \frac{\sqrt{3}}{12}\frac{1-\left(\frac{4}{9}\right)^n}{1-\left(\frac{4}{9}\right)}$$

$$\lim A_n = A_0 + \frac{\sqrt{3}}{12}\sum_0^{\infty}\left(\frac{4}{9}\right)^n = \frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{12}\frac{1}{1-\left(\frac{4}{9}\right)} = \frac{3\sqrt{3}}{12} + \frac{\sqrt{3}}{12}\frac{9}{5} = \frac{\sqrt{3}}{4}\left[1 + \frac{3}{5}\right] = \frac{2\sqrt{3}}{5}$$