Special Series in Geometry (3)
Rieman's Integral's sum

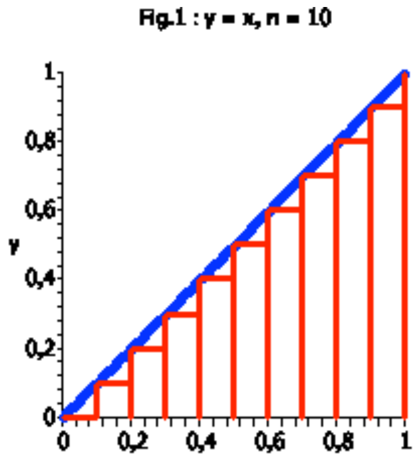
0. Pove the following formulas :

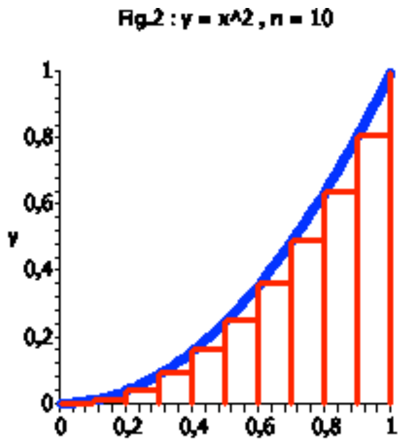
$$S_n = 1+2+3+...+n = \sum_{i=1}^{n} i = \frac{n(n+1)}{2} \quad \because\ 2S_n = n(n+1)\ ; \quad \text{and} \quad T_n = 1^2+2^2+3^2+...+n^2 = \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\left\{\begin{array}{c} (1+1)^3 = 1^3 + 3\times 1^2 \times 1 + 3\times 1^1 \times 1^2 + 1 \\ (2+1)^3 = 2^3 + 3\times 2^2 \times 1 + 3\times 2^1 \times 1^2 + 1 \\ (3+1)^3 = 3^3 + 3\times 3^2 \times 1 + 3\times 3^1 \times 1^2 + 1 \\ \cdots\cdots \\ (n)^3 = (n-1)^3 + 3\times (n-1)^2 \times 1 + 3\times (n-1)^1 \times 1^2 + 1 \\ (n+1)^3 = n^3 + 3\times n^2 \times 1 + 3\times n^1 \times 1^2 + 1 \end{array}\right\} \Rightarrow \left\{\begin{array}{l} \textit{by adding all these equalities} \\ \textit{and reducing the common terms,} \\ \quad\quad \textit{we get :} \\ (n+1)^3 = 1 + 3\sum_{i=1}^{n} i^2 + 3\sum_{i=1}^{n} i + n \end{array}\right\}$$

$$\Leftrightarrow n^3 + 3n^2 + 2n = 3\sum_{i=1}^{n} i^2 + 3\sum_{i=1}^{n} i \Leftrightarrow 3\sum_{i=1}^{n} i^2 = n(n^2+3n+2) - 3\frac{n(n+1)}{2} = \frac{n(n+1)(2n+1)}{2} \Leftrightarrow \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

1. Construction of the basic patterns :





In these two graphics the Interval [0,1] is divided into n = 10 equal parts.

2. Evaluate the area of the rectangles defined by this division into 10 parts :

The area of each rectangle is $\frac{1}{10}\left(\frac{i}{10}\right)$

$$A_{10} = \sum_{i=0}^{i=9} \frac{1}{10}\left(\frac{i}{10}\right) = \frac{1}{100}\sum_{i=0}^{i=9} i = \frac{9\times 10}{200} = 0.45$$

The area of each rectangle is $\frac{1}{10}\left(\frac{i}{10}\right)^2$

$$B_{10} = \sum_{i=0}^{i=9} \frac{1}{10}\left(\frac{i}{10}\right)^2 = \frac{1}{1000}\sum_{i=0}^{i=9} i^2 = \frac{9\times 10\times 19}{6000} = 0.285$$

3. Evaluate the area of the rectangles defined by a division into n equal parts :

$$A_n = \frac{1}{n^2}\sum_{i=0}^{i=n-1} i = \frac{(n-1)n}{2n^2} = \frac{n-1}{2n}$$

$$B_n = \frac{1}{n^3}\sum_{i=0}^{i=n-1} i^2 = \frac{(n-1)n(2n-1)}{6n^3} = \frac{2n^2-3n+1}{6n^2}$$

4. Find the limits A and B of (A_n) and (B_n) :

$$\lim_{n\to\infty} A_n = \lim_{n\to\infty} \frac{n-1}{2n} = \lim_{n\to\infty} \frac{1-\frac{1}{n}}{2} = \frac{1}{2}$$

$$\lim_{n\to\infty} B_n = \lim_{n\to\infty} \frac{2n^2-3n+1}{6n^2} = \lim_{n\to\infty} \frac{2-\frac{3}{n}+\frac{1}{n^2}}{6} = \frac{1}{3}$$

5. *One formula to get the area under the curve of a function f on [0;1] is :* $S = \sum_{i=0}^{\infty} \frac{1}{n} f\left(\frac{i}{n}\right)$