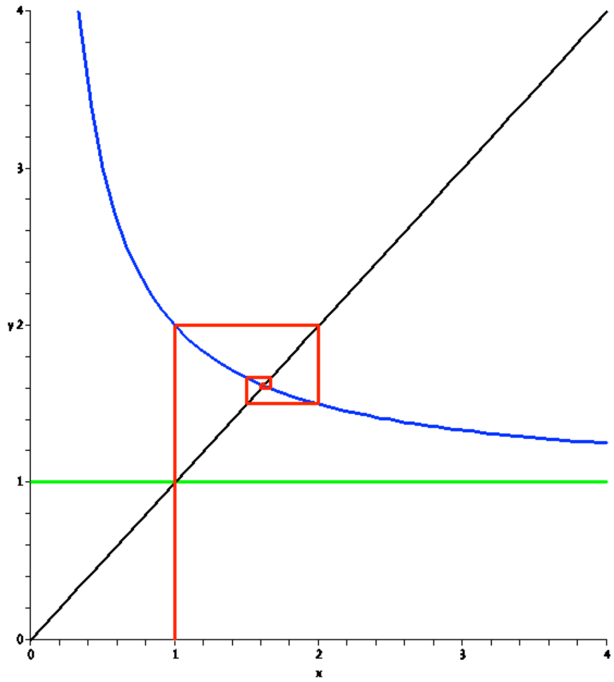
Problems of Convergence of Non Monotonous Recursive Sequences using **Recurrence Reasonning** :

I. Let $f(x) = 1 + \frac{1}{x}$ for x > 0 , (u_n) defined by : $u_{n+1} = 1 + \frac{1}{u_n}$ with $u_0 = 1$.

1. $u_1 = 1 + \frac{1}{1} = 2$; $u_2 = 1 + \frac{1}{1+\frac{1}{1}} = \frac{3}{2} = 1.5$; $u_3 = 1 + \frac{1}{1+\frac{1}{1+\frac{1}{1}}} = \frac{5}{3} = 1.6$;

2. Construction of the first terms of the sequence (u_n).



3. Show by <u>recurrence</u>, that for any n ≥0 , [P_n] $1 \le u_n \le 2$
   i. <u>Initialization</u> : [P_0] ⇔ $1 \le u_0 \le 2$ TRUE
   ii. <u>Heredity</u> : *f is <u>decreasing</u> on [0 ; +∞[hence* $1 \le u_n \le 2 \Rightarrow f(1) \ge f(u_n) \ge f(2)$
   $f(1) = 2$, $f(u_n) = u_{n+1}$ and $f(2)=1.5$ then [P_n] ⇒ [P_{n+1}] *(n fixed).*
   iii. *<u>Conclusion</u>* : by recurrence, [P_n] is true for any n ≥ 0.
4. From the graph indicate whether (u_n) is monotonous : *NO !*
5. From the graph indicate whether (u_n) is converging. If yes, what is the limit ?
   It converges to the abscissa of the interception point between the graph of f and the 1st bissector (y=x).

II. Let $g = f \circ f \Leftrightarrow g(x) = f[f(x)]$

1. $g(x) = 1 + \frac{1}{1+\frac{1}{x}} = \frac{2x+1}{x+1} = 2 - \frac{1}{x+1}$; $g = f \circ f$ is increasing because f is decreasing /

2. Let $v_n = u_{2n}$ and $w_n = u_{2n+1}$

   i. $v_{n+1} = u_{2n+2} = f(u_{2n+1}) = f(f(u_{2n})) = g(v_n)$; $v_0 = u_0 = 1$

   $w_{n+1} = u_{2n+3} = f(u_{2n+2}) = f(f(u_{2n+1})) = g(w_n)$; $w_0 = u_1 = 2$

   ii. Show <u>by recurrence</u>, (v_n) is increasing : let $[P_n]$ $\boxed{v_n \le v_{n+1}}$

- <u>Initialization</u> : $[P_0] \Leftrightarrow v_0 \le v_1$ TRUE ($v_0 = u_0 = 1$ and $v_1 = u_2 = 1.5$)
- <u>Heredity</u> : *g is <u>increasing</u> on [0 ; +∞[hence* $v_n \le v_{n+1} \Rightarrow g(v_n) \le g(v_{n+1})$
  $g(v_n) = v_{n+1}$ and $g(v_{n+1}) = v_{n+2}$ then $[P_n] \Rightarrow [P_{n+1}]$ *(n fixed).*
- <u>Conclusion</u> : by recurrence, $[P_n]$ is true for any $n \ge 0$.

3. Prove that for any $n \ge 0$ $|v_{n+1} - w_{n+1}| \le \frac{1}{4}|v_n - w_n|$ *(for any n, $u_n \ge 1$)*

$$|v_{n+1} - w_{n+1}| \le |g(v_n) - g(w_n)| = \left|\left(2 - \frac{1}{v_n + 1}\right) - \left(2 - \frac{1}{w_n + 1}\right)\right| = \left|\frac{w_n - v_n}{(w_n + 1)(v_n + 1)}\right| \le \frac{1}{4}|w_n - v_n|$$

4. Prove <u>by recurrence</u> that for any $n \ge 0$ $[P_n]$ $\boxed{|v_n - w_n| \le \left(\frac{1}{4}\right)^n}$,

   i. <u>Initialization</u> : $[P_0] \Leftrightarrow |v_0 - w_0| \le \left(\frac{1}{4}\right)^0 \Leftrightarrow |1-2| \le 1$ TRUE !

   ii. <u>Heredity</u> :

$$\left\{\begin{array}{l} |v_{n+1} - w_{n+1}| \le \frac{1}{4}|v_n - w_n| \\ [P_n] \quad |v_n - w_n| \le \left(\frac{1}{4}\right)^n \end{array}\right\} \Rightarrow |v_{n+1} - w_{n+1}| \le \frac{1}{4}\left(\frac{1}{4}\right)^n = \left(\frac{1}{4}\right)^{n+1} \quad [P_{n+1}]$$

   <u>Conclusion</u> : by recurrence, $[P_n]$ is true for any $n \ge 0$.

   iii. $\lim |v_n - w_n| = 0$ because $\lim \left(\frac{1}{4}\right)^n = 0$

   iv. The two sequences (v_n) and (w_n) are <u>adjacent</u>, then have the same limit α.

   From the relationship $u_{n+1}=f(u_n)$ and f is a contiuous function for $x > 0$

$$\lim u_{n+1} = \lim f(u_n) = f(\lim(u_n)) \Leftrightarrow \alpha = f(\alpha) \Leftrightarrow \alpha \textit{ solution of the equation } x = 1 + \frac{1}{x} \ (x > 0) \Leftrightarrow \alpha = \frac{1+\sqrt{5}}{2}$$