Problems of Convergence of Non Monotonous Recursive Sequences using **Recurrence Reasoning** :

Study of the sequence defined by (n consecutive fractions)

$$u_n = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{\cfrac{\cdots\cdots}{1 + \cfrac{1}{1}}}}}}$$

I. Let $f(x) = 1 + \frac{1}{x}$ for x > 0 , (u_n) defined by : $u_{n+1} = 1 + \frac{1}{u_n}$ with $u_0 = 1$.

1. Write u_1 ; u_2 ; u_3 .
2. On the back of the page graph the function f on [0 ; 2] and show the construction of the first terms of the sequence (u_n).
3. Show by recurrence, that for any n ≥0 , $1 \leq u_n \leq 2$
4. From the graph indicate whether (u_n) is monotonous.
5. From the graph indicate whether (u_n) is converging. If yes, what is the limit ?

II. Let $g = f \circ f \Leftrightarrow g(x) = f[f(x)]$

1. Calculate $g(x)$, and show that g is increasing on [0 ; +∞[
2. Let $v_n = u_{2n}$ and $w_n = u_{2n+1}$
   i. Show that $v_{n+1} = g(v_n)$ and that $w_{n+1} = g(w_n)$; give v_0 and w_0
   ii. Show by recurrence, that (w_n) is decreasing and that (v_n) is increasing.
   iii. Prove that for any n ≥ 0 $|v_{n+1} - w_{n+1}| \leq \frac{1}{4}|v_n - w_n|$
   iv. Prove by recurrence that for any n ≥ 0 $|v_n - w_n| \leq \left(\frac{1}{4}\right)^n$,
   v. Prove that $\lim |v_n - w_n| = 0$
   vi. The two sequences (v_n) and (w_n) are adjacent, then have the same limit α.

   Prove that this limit is the positive solution of the equation $x = 1 + \frac{1}{x}$, and give the exact value of α.