Numerical Sequences

Problem I : Let f be the function defined by $f(x)=\dfrac{2x+3}{x+4}$ for x ≥ 0.

Study the Sequence defined by the formula $u_n = f(n) = \dfrac{2n+3}{n+4}$ for every n ∈ N.

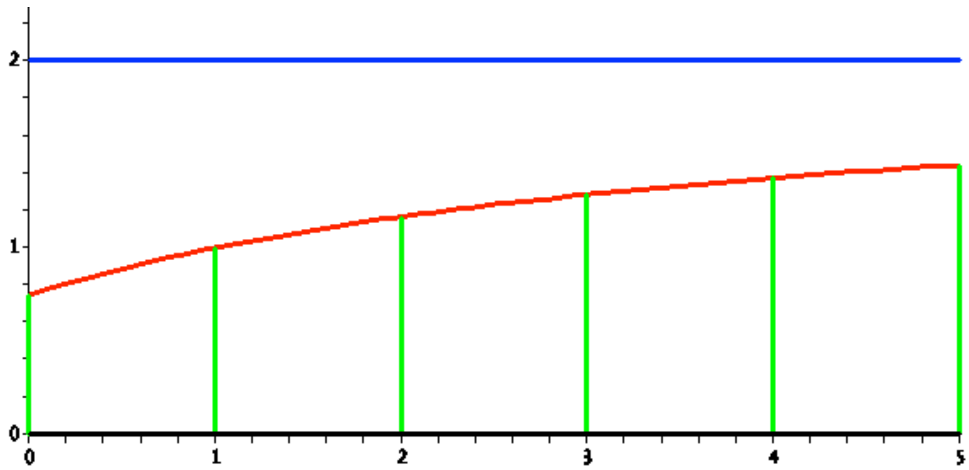
a. Graph the function *f* on [0 ; +∞ [and draw the first terms of the sequence (u_n).
   Indicate from the graph whether or not the sequence is :
   i. Monotonous (if yes how) :
   ii. Bounded (*if yes, what are the boundaries ?)*
   iii. Does-it seem to have a limit (*if yes which one is it?*)?

b. Prove that (u_n) is increasing

c. Prove that (u_n) is bounded by 0 and 2.

d. Find for which value of n we have : 2 - ε < u_n < 2 with ε =10^{-2}

e. Prove that for any n ≥1 we have $|u_n - 2| \le \dfrac{5}{n}$. Conclusion ?



a) *From the graph we can tell :*
(i) (u_n) is increasing; (ii) 0.75 ≤ u_n ≤ 2 ; (iii) Lim(u_n) =2

b) *From the properties of the function f,* $f(x)=\dfrac{2x+3}{x+4}=2-\dfrac{5}{x+4}$ *we can tell that (u_n) is increasing because f is increasing on [0; +∞[and u_n=f(n).*

c) *Because f is increasing on [0; +∞[, f(0) = .75 and lim f(x) = 2 then for any Integer n, 0.75 ≤ u_n ≤ 2.*

d)
$$2-\varepsilon \le u_n \le 2+\varepsilon \Leftrightarrow 2-\varepsilon \le 2-\frac{5}{n+4} \le 2+\varepsilon \Leftrightarrow -\varepsilon \le -\frac{5}{n+4} \le \varepsilon \Leftrightarrow -\varepsilon \le \frac{5}{n+4} \le \varepsilon$$
$$\Leftrightarrow (\varepsilon>0)\ \frac{n+4}{5} \ge \frac{1}{\varepsilon} \Leftrightarrow n \ge \frac{5}{\varepsilon}-4 \Leftrightarrow (\textit{for } \varepsilon = \frac{1}{100})\ n \ge 496$$

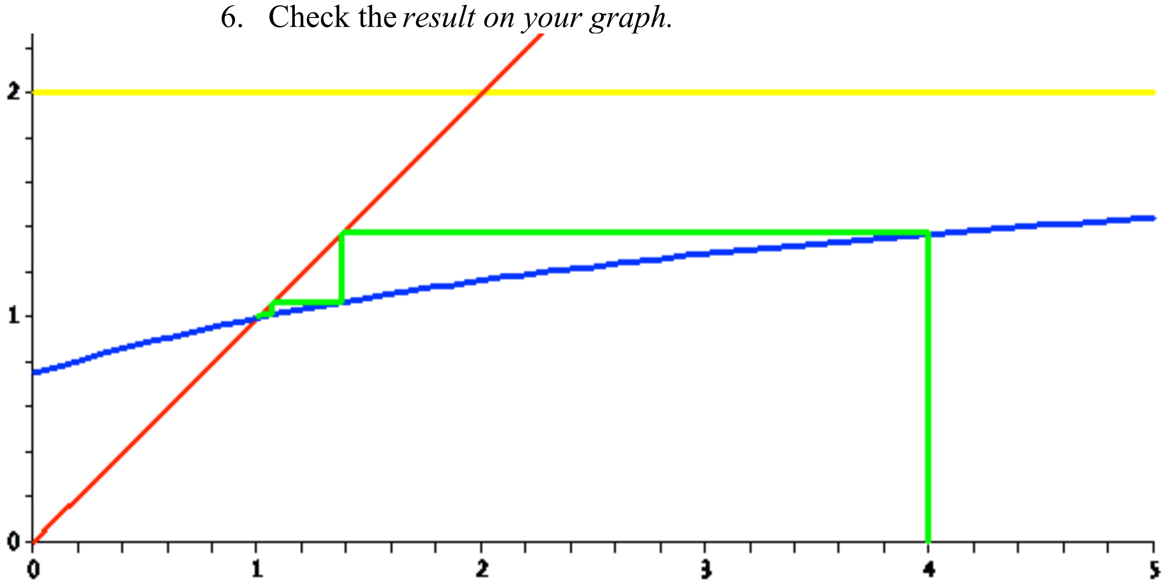
e)
$$|u_n-2| = \left|\frac{-5}{n+4}\right| = \frac{5}{n+4} \le \frac{5}{n} \text{ and } \lim_{n\to+\infty}\left(\frac{5}{n}\right)=0 \text{ then } \lim_{n\to+\infty}|u_n-2|=0$$
which means that $\lim_{n\to+\infty} u_n = 2$

Problem II : Let f be the function defined by $f(x)=\dfrac{2x+3}{x+4}$ for x ≥ 0.

Study of the sequence (v_n) defined by $v_{n+1}=f(v_n)=\dfrac{2v_n+3}{v_n+4}$; n ≥1 and $v_0 = 4$.

1. Graph the function f on [0 ; +∞ [and draw the first terms of the sequence (v_n).
   Find the coordinates of the intersection of (Cf) with the first bisector (y=x)
   Indicate from the graph whether or not the sequence is :
   i. Monotonous (if yes how) :
   ii. Bounded (*if yes, what are the boundaries ?)*
   iii. Does-it seam to have a limit (*if yes which one is it?*)?
2. Let $w_n=\dfrac{v_n-1}{v_n+3}$ *for any n>0.*

   Show that the new sequence (w_n) is a geometric sequence :
   1. Find its first term and its reason.
   2. Find the expression of w_n directly in function of n.
   3. Deduct the limit of w_n.
   4. Find the expression of v_n in function *of* w_n
   5. Find the limit of v_n
   6. Check the *result on your graph.*



1) From the graph of the sequence we can tell that the sequence (v_n) is decreasing, bounded by 1 and 4 and seems to be converging towards 1 (abscissa of the interception of f with the 1st bisector.

2) (w_n) is a geometric sequence because the ratio of two consecutive terms is constant :

$$w_{n+1}=\frac{v_{n+1}-1}{v_{n+1}+3}=\frac{\dfrac{2v_n+3}{v_n+4}-1}{\dfrac{2v_n+3}{v_n+4}+3}=\frac{v_n-1}{5v_n+15}=\frac{1}{5}\cdot\frac{v_n-1}{v_n+3}=\frac{1}{5}w_n \;;\; w_0=\frac{v_0-1}{v_0+3}=\frac{4-1}{4+3}=\frac{3}{7} \text{ and the reason } q=\frac{1}{5}$$

$$\therefore w_n=\frac{3}{7}\left(\frac{1}{5}\right)^n \Rightarrow \lim_{n\to+\infty} w_n=0 \;(\because\; |q|<1)$$

$$w_n=\frac{v_n-1}{v_n+3}\Leftrightarrow v_n=\frac{1+3w_n}{1-w_n} \text{ then } \lim_{n\to+\infty} v_n=\frac{1+3\times 0}{1-0}=1$$

The interception point of the graph of f and of the 1st bisector (y=x) yields the equation

$$\frac{2x+3}{x+4}=x \text{ and } x>0 \Leftrightarrow x^2+2x-3=0 \text{ and } x>0 \Leftrightarrow x=1$$