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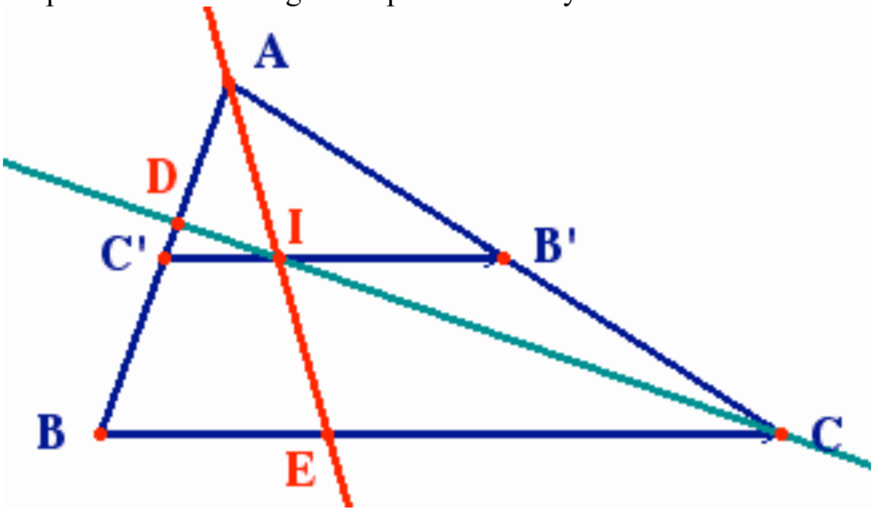
Problem (VSOP*) :** Let ABC be a normal triangle, B' the middle of AC and C' the middle of AB. We want to place 3 special points on this triangle and prove that they are on the same line.



1. Determine and place the point I defined by the vector equation : $2\overrightarrow{IC'}+\overrightarrow{IB'}=\vec{O}$
2. Determine and place the point D defined by the vector equation : $3\overrightarrow{DA}+2\overrightarrow{DB}=\vec{O}$
3. Determine and place the point E defined by the vector equation : $2\overrightarrow{EB}+\overrightarrow{EC}=\vec{O}$
4. Prove by using a vector equation that the 3 points **A, I, E**, are on the same line.
5. Prove by using a vector equation that the 3 points **C, I, D,** are on the same line.

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1. $2\overrightarrow{IC'}+\overrightarrow{IB'}=\vec{O}\Leftrightarrow 2(\overrightarrow{IB'}+\overrightarrow{B'C'})+\overrightarrow{IB'}=\vec{O}\Leftrightarrow 3\overrightarrow{IB'}+2\overrightarrow{B'C'}=\vec{O}\Leftrightarrow 3\overrightarrow{B'I}=2\overrightarrow{B'C'}\Leftrightarrow \overrightarrow{B'I}=\frac{2}{3}\overrightarrow{B'C'}$
2. $3\overrightarrow{DA}+2\overrightarrow{DB}=\vec{O}\Leftrightarrow 3\overrightarrow{DA}+2(\overrightarrow{DA}+\overrightarrow{AB})=\vec{O}\Leftrightarrow 5\overrightarrow{DA}+2\overrightarrow{AB}=\vec{O}\Leftrightarrow 5\overrightarrow{AD}=2\overrightarrow{AB}\Leftrightarrow \overrightarrow{AD}=\frac{2}{5}\overrightarrow{AB}$
3. $2\overrightarrow{EB}+\overrightarrow{EC}=\vec{O}\Leftrightarrow 2\overrightarrow{EB}+\overrightarrow{EB}+\overrightarrow{BC}=\vec{O}\Leftrightarrow 3\overrightarrow{EB}+\overrightarrow{BC}=\vec{O}\Leftrightarrow 3\overrightarrow{BE}=\overrightarrow{BC}\Leftrightarrow \overrightarrow{BE}=\frac{1}{3}\overrightarrow{BC}$
4. To prove that the 3 points are on a same line we must find a number k such that $\overrightarrow{AI}=k\overrightarrow{AE}$
   To do so we can find $\overrightarrow{AI}=x\overrightarrow{AB}+y\overrightarrow{AC}$ and $\overrightarrow{AE}=x'\overrightarrow{AB}+y'\overrightarrow{AC}$ then prove that $x=k.x'$ and $y=k.y'$

   From (1) $2\overrightarrow{IC'}+\overrightarrow{IB'}=\vec{O}$ we get

   $$2(\overrightarrow{IA}+\overrightarrow{AC'})+(\overrightarrow{IA}+\overrightarrow{AB'})=\vec{O}\Leftrightarrow 3\overrightarrow{IA}+2\overrightarrow{AC'}+\overrightarrow{AB'}=\vec{O}\Leftrightarrow 3\overrightarrow{AI}=\overrightarrow{AB}+\frac{1}{2}\overrightarrow{AC}\Leftrightarrow \overrightarrow{AI}=\frac{1}{3}\overrightarrow{AB}+\frac{1}{6}\overrightarrow{AC}$$

   From (3) $2\overrightarrow{EB}+\overrightarrow{EC}=\vec{O}$ we get

   $$2(\overrightarrow{EA}+\overrightarrow{AB})+(\overrightarrow{EA}+\overrightarrow{AC})=\vec{O}\Leftrightarrow 3\overrightarrow{EA}+2\overrightarrow{AB}+\overrightarrow{AC}=\vec{O}\Leftrightarrow \overrightarrow{AE}=\frac{2}{3}\overrightarrow{AB}+\frac{1}{3}\overrightarrow{AC}$$

   Hence we have $\overrightarrow{AI}=\frac{1}{2}\overrightarrow{AE}$ which proves that A,I,E are on the same line (I is the midpoint of [AE])
5. Same method : let's prove that $\overrightarrow{ID}=k\overrightarrow{IC}$ by de-composing and recomposing …

   We have $\overrightarrow{IC}=\overrightarrow{IA}+\overrightarrow{AC}=\left(-\frac{1}{3}\overrightarrow{AB}-\frac{1}{6}\overrightarrow{AC}\right)+\overrightarrow{AC}=-\frac{1}{3}\overrightarrow{AB}+\frac{5}{6}\overrightarrow{AC}$

   We also have $\overrightarrow{ID}=\overrightarrow{IA}+\overrightarrow{AD}=\left(-\frac{1}{3}\overrightarrow{AB}-\frac{1}{6}\overrightarrow{AC}\right)+\frac{2}{5}\overrightarrow{AB}=\frac{1}{15}\overrightarrow{AB}-\frac{1}{6}\overrightarrow{AC}$

   Hence we have $\overrightarrow{ID}=-\frac{1}{5}\overrightarrow{IC}$ which proves that D,I,C are on the same line.