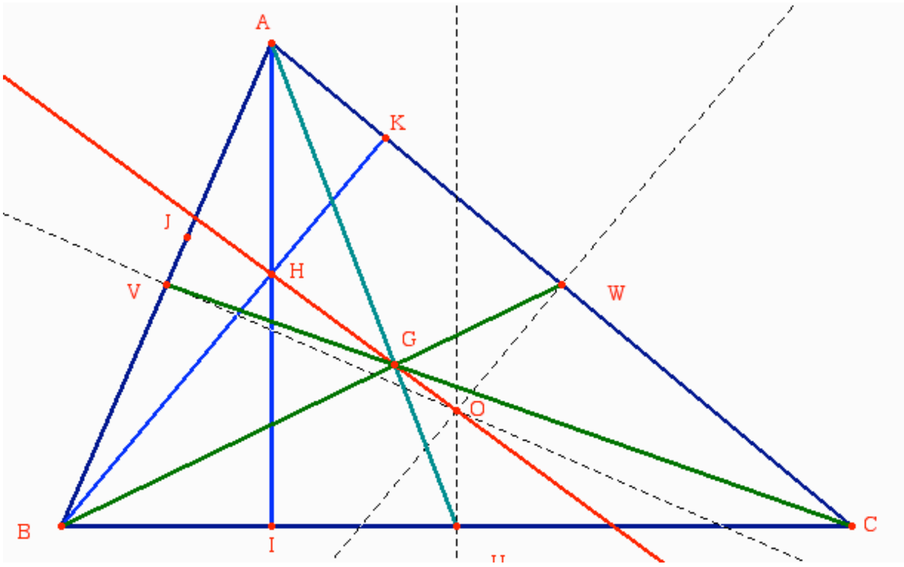
北京景山学校 Name : 纪老师
jiguanglaoshi@gmail.com
http://beijingshanmaths.org

Euler's line proof by using vector calculations only.

Let ABC be a regular triangle, U the midpoint of BC, V the midpoint of AB, and W the midpoint of AC.



1. Let M be the point defined by the vectorial equation $\overrightarrow{MA}+\overrightarrow{MB}+\overrightarrow{MC}=\vec{O}$
   Show WHY this point M is the intersection G of the three median lines AU,BV,CW.

2. Let N be the point defined by the vectorial equation $\overrightarrow{OA}+\overrightarrow{OB}+\overrightarrow{OC}=\overrightarrow{ON}$
   Show WHY this point is the intersection H of the three heights AI, BK, CJ

3. From the above vectorial relations prove that O, G, H are on the same line (Euler's line).

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1. We can write : $\overrightarrow{MA}+\overrightarrow{MB}+\overrightarrow{MC}=\vec{O} \Leftrightarrow 3\overrightarrow{MA}+\overrightarrow{AB}+\overrightarrow{AC}=\vec{O} \Leftrightarrow 3\overrightarrow{MA}+2\overrightarrow{AU}=\vec{O} \Leftrightarrow \overrightarrow{AM}=\frac{2}{3}\overrightarrow{AU}$

   Similarly $\overrightarrow{MA}+\overrightarrow{MB}+\overrightarrow{MC}=\vec{O} \Leftrightarrow 3\overrightarrow{MB}+\overrightarrow{BA}+\overrightarrow{BC}=\vec{O} \Leftrightarrow 3\overrightarrow{MB}+2\overrightarrow{BW}=\vec{O} \Leftrightarrow \overrightarrow{BM}=\frac{2}{3}\overrightarrow{BW}$

   and $\overrightarrow{MA}+\overrightarrow{MB}+\overrightarrow{MC}=\vec{O} \Leftrightarrow 3\overrightarrow{MC}+\overrightarrow{CA}+\overrightarrow{CB}=\vec{O} \Leftrightarrow 3\overrightarrow{MC}+2\overrightarrow{CV}=\vec{O} \Leftrightarrow \overrightarrow{CM}=\frac{2}{3}\overrightarrow{CV}$

   Finally M belongs to the three medians AU, BW, and CV, so by definition M=G (c. of Gravity).

2. We can write : $\overrightarrow{OA}+\overrightarrow{OB}+\overrightarrow{OC}=\overrightarrow{ON} \Leftrightarrow \overrightarrow{OA}+2\overrightarrow{OU}=\overrightarrow{ON} \Leftrightarrow 2\overrightarrow{OU}=\overrightarrow{AO}+\overrightarrow{ON} \Leftrightarrow 2\overrightarrow{OU}=\overrightarrow{AN}$
   hence the line AN is parallel to OU therefore AN is the height AI of the triangle ABC.
   Similarly $\overrightarrow{OA}+\overrightarrow{OB}+\overrightarrow{OC}=\overrightarrow{ON} \Leftrightarrow \overrightarrow{OB}+2\overrightarrow{OW}=\overrightarrow{ON} \Leftrightarrow 2\overrightarrow{OW}=\overrightarrow{BO}+\overrightarrow{ON} \Leftrightarrow 2\overrightarrow{OW}=\overrightarrow{BN}$
   hence the line BN is parallel to OW therefore BN is the height BK of the triangle ABC.
   and $\overrightarrow{OA}+\overrightarrow{OB}+\overrightarrow{OC}=\overrightarrow{ON} \Leftrightarrow 2\overrightarrow{OV}+\overrightarrow{OC}=\overrightarrow{ON} \Leftrightarrow 2\overrightarrow{OV}=\overrightarrow{CO}+\overrightarrow{ON} \Leftrightarrow 2\overrightarrow{OV}=\overrightarrow{CN}$
   hence the line CN is parallel to OV therefore BN is the height CJ of the triangle ABC.
   Finally N belongs to the three heights, so by definition N = H orthocenter of the triangle ABC

3. We have now : $\overrightarrow{GA}+\overrightarrow{GB}+\overrightarrow{GC}=\vec{O}$ and $\overrightarrow{OA}+\overrightarrow{OB}+\overrightarrow{OC}=\overrightarrow{OH}$

   then $\overrightarrow{GA}+\overrightarrow{GB}+\overrightarrow{GC}=\vec{O} \Leftrightarrow 3\overrightarrow{GO}+\overrightarrow{OA}+\overrightarrow{OB}+\overrightarrow{OC}=\vec{O} \Leftrightarrow 3\overrightarrow{GO}+\overrightarrow{OH}=\vec{O} \Leftrightarrow \overrightarrow{OG}=\frac{1}{3}\overrightarrow{OH}$

   which is proving that the three points O,G,H are on the same line.