Numerical Sequences

Problem I : Let f be the function defined by $f(x)=\dfrac{2x+3}{x+4}$ for $x \geq 0$.

Study the Sequence defined by the formula $\boxed{u_n = f(n) = \dfrac{2n+3}{n+4}}$ for every $n \in \mathbb{N}$.

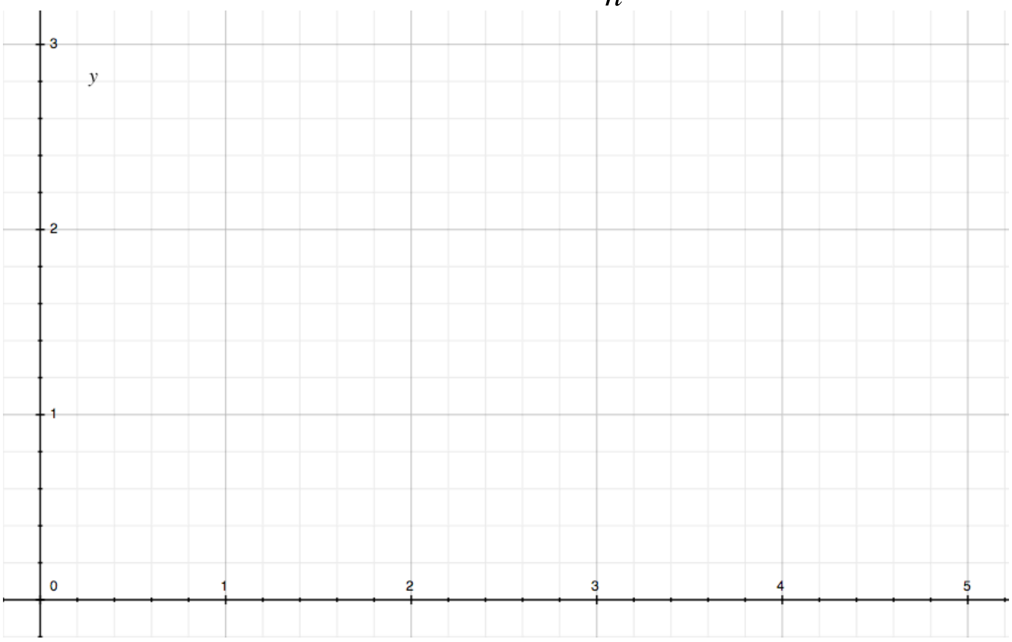
a. Graph the function f on $[0\ ;+\infty[$ and draw the first terms of the sequence (u_n). Indicate from the graph whether or not the sequence is :
   i. Monotonous (if yes how) :
   ii. Bounded (*if yes, what are the boundaries ?)*
   iii. Does-it seem to have a limit (*if yes which one is it?*)?

b. Prove that (u_n) is increasing

c. Prove that (u_n) is bounded by 0 and 2.

d. Find for which value of n we have : $2 - \varepsilon < u_n < 2$ with $\varepsilon = 10^{-2}$

e. Prove that for any $n \geq 1$ we have $|u_n - 2| \leq \dfrac{5}{n}$. Conclusion ?



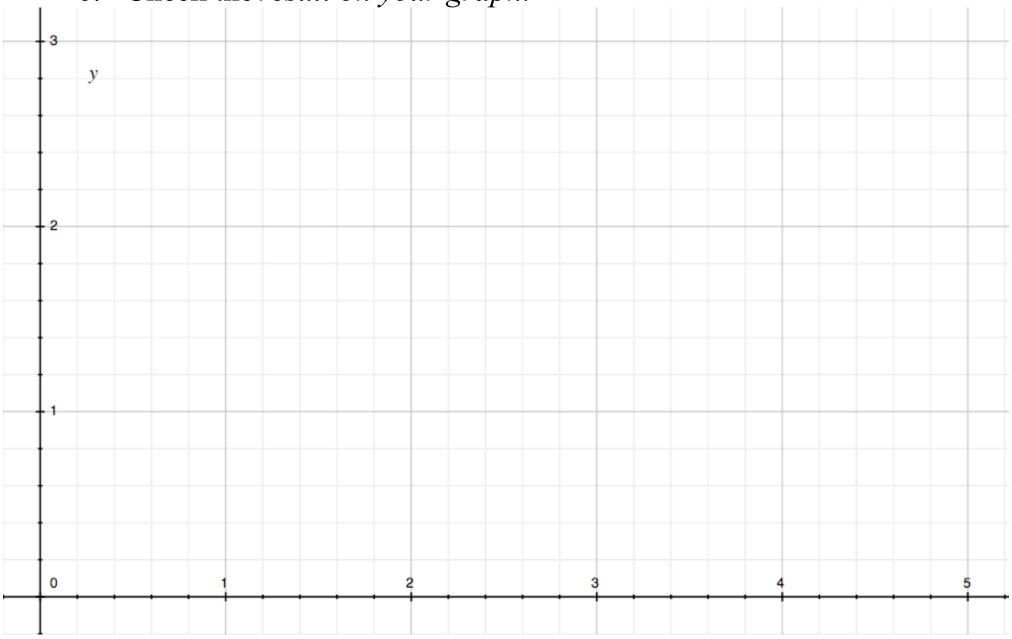
Problem II : Let f be the function defined by $f(x)=\dfrac{2x+3}{x+4}$ for $x \geq 0$.

Study of the sequence (v_n) defined by $v_{n+1}=f(v_n)=\dfrac{2v_n+3}{v_n+4}$; $n \geq 1$ and $v_0 = 4$.

1. Graph the function f on [0 ; +∞ [and draw the first terms of the sequence (u_n).
   Find the coordinates of the intersection of (Cf) with the first bisector (y=x)
   Indicate from the graph whether or not the sequence is :
   i. Monotonous (if yes how) :
   ii. Bounded (*if yes, what are the boundaries ?)*
   iii. Does-it seam to have a limit (*if yes which one is it?*)?

2. Let $w_n = \dfrac{v_n - 1}{v_n + 3}$ *for any n>0.*

   Show that the new sequence (w_n) is a geometric sequence :
   1. Find its first term and its reason.
   2. Find the expression of w_n directly in function of n.
   3. Deduct the limit of w_n.
   4. Find the expression of v_n in function *of* w_n
   5. Find the limit of v_n
   6. Check the *result on your graph.*



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