$\boxed{\text{I}}$ $\boxed{f(x) = \dfrac{x}{1+x^2}}$; $\boxed{g(x) = \dfrac{x^3}{1+x^2}}$ Co/Cg. 28.01.08 (1)/(2)

① $I_1 = \int_0^1 f(x)\,dx = F(1) - F(0)$ avec $F(x)$ PRIMITIVE de $f(x)$.

$f(x) = \frac{1}{2}\left(\frac{2x}{1+x^2}\right)$ Forme $k.\frac{u'}{u}$ $\Rightarrow F(x) = k \ln|u| = \frac{1}{2} \times \ln(1+x^2)$

$\Rightarrow I_1 = \frac{1}{2}\ln(1+1) - \frac{1}{2}\ln 1 = \boxed{\frac{1}{2}\ln 2}$ car $\ln 1 = 0$.

② $I_2 = \int_0^1 g(x)\,dx \Rightarrow I_1 + I_2 = \int_0^1 \left[\frac{x}{1+x^2} + \frac{x^3}{1+x^2}\right]dx$

$\Rightarrow I_1 + I_2 = \int_0^1 \frac{x+x^3}{1+x^2}dx = \int_0^1 \frac{x(1+x^2)}{1+x^2}dx = \int_0^1 x\,dx = \left[\frac{x^2}{2}\right]_0^1 = \boxed{\frac{1}{2}}$

Par suite $I_2 = \frac{1}{2} - I_1 = \boxed{\frac{1}{2} - \frac{1}{2}\ln 2}$ ou $\frac{1 - \ln 2}{2}$

II ① $\boxed{g(x) = \dfrac{1}{x(x^2-1)}}$ $x > 1$ a) $g(x) = \frac{a}{x} + \frac{b}{x+1} + \frac{c}{x-1} = \frac{a(x^2-1) + b(x^2+x) + c(x^2-x)}{x(x^2-1)}$.

Donc $\frac{1}{x(x^2-1)} = \frac{(a+b+c)x^2 + (b-c)x - a}{x(x^2-1)}$

Par identification on a donc $\begin{cases} a+b+c = 0 \\ b-c = 0 \\ -a = 1 \end{cases} \Leftrightarrow \begin{cases} a = -1 \\ b = c = \frac{1}{2} \end{cases}$

$\Rightarrow g(x) = -\frac{1}{x} + \frac{1}{2}\frac{1}{x+1} + \frac{1}{2}\frac{1}{x-1}$ $[x > 1]$.

b) $G(x) = -\ln x + \frac{1}{2}\ln(x+1) + \frac{1}{2}\ln(x-1) = \frac{1}{2}\ln(x^2-1) - \ln x$ $(+k)$

② $f(x) = \frac{2x}{(x^2-1)^2}$ FORME : $+\frac{u'}{u^2}$ $\Rightarrow F(x) = -\frac{1}{x^2-1} \cdot (+k) \cdot (x > 1)$

③ $I = \int_2^3 \frac{2x}{(x^2-1)^2}\ln x = \left[-\frac{\ln x}{x^2-1}\right]_2^3 + \int_2^3 \frac{dx}{x(x^2-1)}$

$-\frac{1}{x^2-1}$; $\frac{1}{x}$.

$= \left[-\frac{\ln x}{x^2-1} + \frac{1}{2}\ln(x^2-1) - \ln x\right]_2^3$

$\Rightarrow I = \left(-\frac{\ln 3}{8} + \frac{1}{2}\ln 8 - \ln 3\right) - \left(-\frac{\ln 2}{3} + \frac{1}{2}\ln 3 - \ln 2\right)$

$= \left(-\frac{1}{8} - 1 - \frac{1}{2}\right)\ln 3 + \left(\frac{3}{2} + \frac{1}{3} + 1\right)\ln 2$ NB: $\ln 8 = \ln 2^3 = 3\ln 2$

$= \boxed{\frac{17}{6}\ln 2 - \frac{13}{8}\ln 3}$

III ① $\dfrac{1}{x(1+x)^2} = \dfrac{1}{x} - \dfrac{1}{1+x} - \dfrac{1}{(1+x)^2}$ $\boxed{x>0}$

② a) $\displaystyle\int_1^X \frac{dx}{x(1+x)^2} = \int_1^X \left(\frac{1}{x}\right)dx - \int_1^X \frac{dx}{1+x} + \int_1^X \frac{-1}{(1+x)^2}dx$

$= \left[\ln x - \ln(1+x) + \dfrac{1}{1+x}\right]_1^X = \left(\ln X - \ln(1+X) + \dfrac{1}{1+X}\right) - \left(\ln 1 - \ln 2 + \dfrac{1}{2}\right)$

$= \boxed{\ln\dfrac{X}{1+X} + \dfrac{1}{1+X} + \ln 2 - \dfrac{1}{2}}$ $(X \geq 1)$.

b) $\displaystyle\varphi(X) = \int_1^X \frac{\ln x}{(1+x)^3}dx = \int_1^X \ln x \times \frac{1}{(1+x)^3}dx$

$u = \ln x \quad u' = \dfrac{1}{x}$

$v' = \dfrac{1}{(1+x)^3} = (1+x)^{-3} \Rightarrow v = \dfrac{(1+x)^{-3+1}}{-3+1} = -\dfrac{1}{2(1+x)^2}$ $\Bigg\} \Rightarrow -u'v = \dfrac{1}{2}\,\dfrac{1}{x(1+x)^2}$

$\displaystyle\Rightarrow \varphi(X) = \left[\frac{-\ln x}{2(1+x)^2}\right]_1^X + \frac{1}{2}\int_1^X \frac{dx}{x(1+x)^2} = \frac{-\ln X}{2(1+X)^2} + \frac{1}{2}\left[\ln\frac{X}{1+X} + \frac{1}{1+X} + \ln 2 - \frac{1}{2}\right]$

c) $\dfrac{\ln X}{(1+X)^2} = \dfrac{\ln X}{X} \cdot \dfrac{X}{(1+X)^2} \longrightarrow 0 \times 0 = 0$ qd $X \to +\infty$. $\left|\begin{array}{l}\lim_{+\infty} \frac{\ln X}{X} = 0 \\ \lim_{+\infty} \frac{X}{X^2} = 0\end{array}\right.$

$\lim_{+\infty} \dfrac{X}{1+X} = \lim_{+\infty} \dfrac{X}{X} = 1 \Rightarrow \lim_{+\infty} \ln\left(\dfrac{X}{1+X}\right) = \ln 1 = 0 \Rightarrow \lim_{+\infty} \varphi(X) = \dfrac{1}{2}\left(\ln 2 - \dfrac{1}{2}\right)$ CQFD

IV 1a) $\displaystyle I(\alpha) = \int_\alpha^1 \frac{1}{t^2} e^{-\frac{1}{t}} dt = \left[e^{-\frac{1}{t}}\right]_\alpha^1 = e^{-1} - e^{-\frac{1}{\alpha}} = \frac{1}{e} - \frac{1}{e^{\frac{1}{\alpha}}}$

• forme $u'e^u = [e^u]'$.

b) $\lim\limits_{\alpha \to 0} I(\alpha) = \boxed{\dfrac{1}{e}}$ car $\dfrac{1}{\alpha} \to +\infty$ $e^{\frac{1}{\alpha}} \to +\infty \Rightarrow \dfrac{1}{e^{1/\alpha}} \to 0$.

② $\displaystyle J(\alpha) = \int_\alpha^1 \frac{1}{t^3} e^{-\frac{1}{t}} dt = \int_\alpha^1 \frac{1}{t} \times \left(\frac{1}{t^2} e^{-\frac{1}{t}}\right) dt = \left[\frac{e^{-\frac{1}{t}}}{t}\right]_\alpha^1 + \int_\alpha^1 \frac{1}{t^2} e^{-\frac{1}{t}} dt$

$-\dfrac{1}{t^2}$ $\quad e^{-\frac{1}{t}}$

$\Rightarrow J(\alpha) = e^{-1} - \dfrac{e^{-\frac{1}{\alpha}}}{\alpha} + I(\alpha) = \boxed{\dfrac{2}{e} - \dfrac{1}{e^{1/\alpha}}\left(\dfrac{1}{\alpha} + 1\right)} = \dfrac{2}{e} - \dfrac{\frac{1}{\alpha}}{e^{\frac{1}{\alpha}}} - \dfrac{1}{e^{1/\alpha}}$.

b) $\lim\limits_{\substack{\alpha \to 0 \\ (\alpha > 0)}} J(\alpha) = \boxed{\dfrac{2}{e}}$ car en posant $X = \dfrac{1}{\alpha}$ on a $\dfrac{\frac{1}{\alpha}}{e^{\frac{1}{\alpha}}} = \dfrac{X}{e^X} \longrightarrow 0$.

$\alpha \to 0^+ \Rightarrow X \to +\infty$.

et on a déjà montré que $\lim\limits_{\alpha \to 0} \dfrac{1}{e^{1/\alpha}} = 0$.
