I - TERRACHER N°92 p. 195 | Nb COMPLEXES | | TS4 / Co - Dev. N°2 / 17.09.04

①/ 2

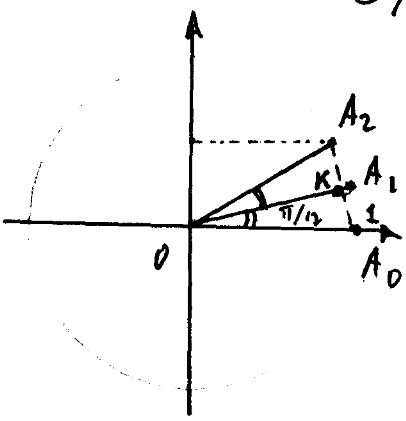
$z_0 = 1 \longleftrightarrow A_0$

$z_1 = e^{i\frac{\pi}{12}} \longleftrightarrow A_1$

$z_2 = e^{i\frac{\pi}{6}} \longleftrightarrow A_2$

K milieu de $[A_0A_2] \Leftrightarrow \vec{OK} = \frac{1}{2}(\vec{OA_0} + \vec{OA_2})$

$\Leftrightarrow z_K = \frac{1}{2}(z_0 + z_2) = \frac{1}{2}(1 + e^{i\frac{\pi}{6}})$



1) $e^{i\frac{\pi}{6}} = \cos\frac{\pi}{6} + i\sin\frac{\pi}{6} = \frac{\sqrt{3}}{2} + i\cdot\frac{1}{2} \Rightarrow \boxed{z_K = \frac{2+\sqrt{3}}{4} + i\cdot\frac{1}{4}}$

2) Dans le triangle OKA_2 rectangle en K on a $OK = \cos\frac{\pi}{12}$

et $(\vec{OA_0}; \vec{OK}) = (\vec{OA_0}; \vec{OA_1}) = \operatorname{Arg}(z_1) = \frac{\pi}{12}$

$\Rightarrow \boxed{z_K = \left(\cos\frac{\pi}{12}\right) e^{i\frac{\pi}{12}}} = \left(\cos\frac{\pi}{12}\right)\left[\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}\right] = \cos^2\frac{\pi}{12} + i\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}$

3°) $(\sqrt{6}+\sqrt{2})^2 = 6 + 2\sqrt{12} + 2 = 8 + 4\sqrt{3} \Rightarrow \sqrt{6}+\sqrt{2} = \sqrt{8+4\sqrt{3}}$

$z_K = \cos^2\frac{\pi}{12} + i\,\frac{1}{2}\sin\frac{\pi}{6} = \frac{2+\sqrt{3}}{4} + i\,\frac{1}{4} \Longleftrightarrow \begin{cases} \cos^2\frac{\pi}{12} = \frac{2+\sqrt{3}}{4} & (1) \\ \frac{1}{2}\sin\frac{\pi}{6} = \frac{1}{4} \end{cases}$

$\Rightarrow \cos\frac{\pi}{12} = \frac{\sqrt{2+\sqrt{3}}}{2} = \frac{2\sqrt{2+\sqrt{3}}}{4} = \frac{\sqrt{8+4\sqrt{3}}}{4} = \boxed{\frac{\sqrt{6}+\sqrt{2}}{4}}$ CQFD.

et $\sin\frac{\pi}{12} = \sqrt{1-\cos^2\frac{\pi}{12}} = \sqrt{1 - \frac{8+4\sqrt{3}}{16}} = \sqrt{\frac{8-4\sqrt{3}}{16}} = \frac{\sqrt{8-4\sqrt{3}}}{4} = \boxed{\frac{\sqrt{6}-\sqrt{2}}{4}}$ CQFD

II. N°107 p. 197 $\quad \omega = e^{2i\frac{\pi}{5}}$

1°) $\omega^5 = \left(e^{2i\frac{\pi}{5}}\right)^5 = e^{2i\pi} = 1$; $1+\omega+\omega^2+\omega^3+\omega^4 = \frac{1-\omega^5}{1-\omega} = 0$.

2°) $u+v = (\omega+\omega^4) + (\omega^2+\omega^3) = -1$

$uv = (\omega+\omega^4)(\omega^2+\omega^3) = \omega^3+\omega^6+\omega^4+\omega^7 = \omega^3+\omega+\omega^4+\omega^2 = -1$

Donc u et v sont sol de l'équation $Z^2 - SZ + P = 0$

avec $S = u+v = -1$ et $P = uv = -1$ $\qquad Z^2+Z-1 = 0$ dont les racines

sont $\frac{-1+\sqrt{5}}{2}$ et $\frac{-1-\sqrt{5}}{2}$

3°) $u = \omega + \omega^4$ mais $\omega^4 = \frac{\omega^5}{\omega} = \frac{1}{\omega} = \bar{\omega}$ car $\omega\bar{\omega} = |\omega|^2 = 1$.

Donc $u = \omega + \bar{\omega} = 2\operatorname{Re}[\omega] = 2\left[\cos\frac{2\pi}{5}\right] > 0$.

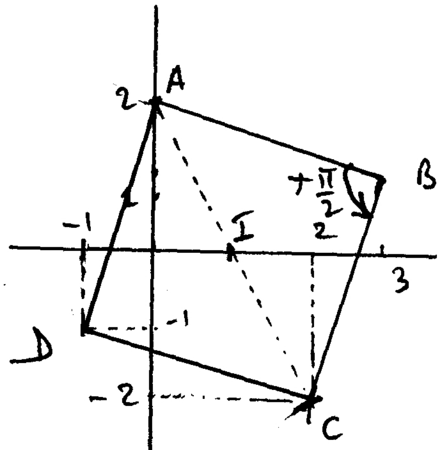
$\Rightarrow u = \frac{-1+\sqrt{5}}{2} = 2\cos\frac{2\pi}{5} \Longleftrightarrow \boxed{\cos\frac{2\pi}{5} = \frac{-1+\sqrt{5}}{4}.}$

(On peut vérifier ce résultat avec une calculatrice).

- $\sin\frac{2\pi}{5} = \sqrt{1-\cos^2\frac{2\pi}{5}} = \sqrt{1-\left(\frac{-1+\sqrt{5}}{4}\right)^2} = \sqrt{\frac{16-(6-2\sqrt{5})}{16}} = \frac{\sqrt{10+2\sqrt{5}}}{4}$

TSVP →

III - 123 p. 200 | $z_I = 1$; $z_A = 2i$; $z_B = 3+i$



1°) $\vec{IC} = -\vec{IA} \iff z_C - z_I = -(z_A - z_I)$

$\iff z_C = z_I - z_A + z_I = \boxed{2-2i}$

2°) $\dfrac{z_C - z_B}{z_A - z_B} = \dfrac{2-2i-3-i}{2i-3-i} = \dfrac{-1-3i}{i-3} = \dfrac{1+3i}{3-i} = \dfrac{(1+3i)(3+i)}{9+1} = \dfrac{10i}{10} = \boxed{i}$

$\iff \dfrac{z_B - z_C}{z_B - z_A} = i = e^{i\frac{\pi}{2}} \iff \left\{ \begin{array}{l} \frac{CB}{BA} = 1 \\ (\vec{BA}, \vec{BC}) = +\frac{\pi}{2} \end{array} \right\} \iff \left| \begin{array}{l} CB = BA \\ \widehat{ABC} \text{ Droit.} \end{array} \right.$

3°) $z_D - z_C = z_A - z_B \iff z_D = z_C + z_A - z_B = 2-2i+2i-3-i = \boxed{-1-i}$

$\Updownarrow$

$\vec{CD} = \vec{BA} \iff$ (ABCD) est un parallélogramme possédant un angle droit et 2 côtés consécutifs égaux $\Rightarrow$ CARRÉ CQFD.

IV - 127 p. 201 $P(z) = z^4 - 6z^3 + 24z^2 - 18z + 63$

$P(i\sqrt{3}) = 0$ et $P(-i\sqrt{3}) \Rightarrow P$ divisible par $(z - i\sqrt{3})$

et par $(z + i\sqrt{3}) \Rightarrow P(z) = (z - i\sqrt{3})(z + i\sqrt{3}) \times Q(z) = (z^2+3)\,Q(z)$

Par division euclidienne de polynômes on obtient :

$Q(z) = z^2 - 6z + 21$ dont les racines complexes sont $\left| \begin{array}{l} 3+2i\sqrt{3} \\ \text{et} \\ 3-2i\sqrt{3} \end{array} \right.$

$z_A = i\sqrt{3}$; $z_B = \overline{z_A}$; $z_C = 3+2i\sqrt{3}$; $z_D = \overline{z_C} \Rightarrow$ le centre du cercle est sur $\vec{Ox}$.

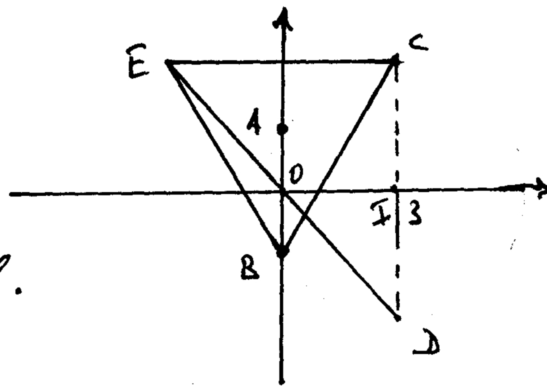
Soit I le point d'affixe 3 ; on a $\left\{ \begin{array}{l} IA = |z_A - z_I| = |i\sqrt{3} - 3| = \sqrt{12} = 2\sqrt{3} \\ IC = |z_C - z_I| = |2i\sqrt{3}| = 2\sqrt{3}. \end{array} \right.$

Donc par symétrie $\left| \begin{array}{l} IB = IA \\ ID = IC \end{array} \right.$

ABCD sont donc sur le cercle de centre I(3;0) et de Rayon $2\sqrt{3}$.

4°) $z_E = -z_D = -3+2i\sqrt{3}$

$\dfrac{z_C - z_B}{z_E - z_B} = \dfrac{(3+2i\sqrt{3}) + i\sqrt{3}}{(-3+2i\sqrt{3}) + i\sqrt{3}} = \dots = \dfrac{-(1+i\sqrt{3})^2}{4} = \dfrac{1}{2} - i\dfrac{\sqrt{3}}{2} = e^{-i\frac{\pi}{3}}$



$\iff \left\{ \begin{array}{l} BC = BE \\ \widehat{\vec{BE}; \vec{BC}} = -\frac{\pi}{3} \end{array} \right\} \iff \widehat{BEC}$ Triangle Equilatéral.